Final Version C

CS540

August 11, 2019

1 Instruction

1. Each incorrect answer receives -0.25, each correct answer receives 1, blank answers receive 0.
2. Check to make sure your name and (numerical) student ID (if you have it) is on the (Scantron) answer sheet. Also, write your Wisc email ID on the answer sheet.
3. Check to make sure you completed question 41 and 42.
4. If you think none (or more than one) of the answers are correct, choose the best (closest) one.
5. Please submit this final, the answer sheet, the formula sheet, and all your additional notes when you finish.
6. Good luck!

①

	AB	C	D	EF
AB	0	7	8	10
C		0	4	6
D			0	3
EF				0

$\Rightarrow$

	AB	C	DEF
AB	0	7	10
C		0	6
DEF			0

②

	A	B	C	DEF
A	0	3	4	7
B		0	7	10
C			0	6
DEF				0

$\Rightarrow$

	AB	C	DEF
AB	0	7	10
C		0	6
DEF			0

2 Questions

1. Given the table summarizing the distances between clusters A to F so far, what is the new cluster after the first step using hierachical clustering with complete-linkage, Manhattan distance, between the clusters.

dist(AB, EF) = max{(A, EF), (B, EF)} = max{7, 10}

–	A	B	C	D	E	F
A	0	3	4	5	6	7
B	–	0	7	8	9	10
C	–	–	0	4	5	6
D	–	–	–	0	2	3
E	–	–	–	–	0	1
F	–	–	–	–	–	0

$\Rightarrow$

	A	B	C	D	EF
A	0	3	4	5	7
B		0	7	8	10
C			0	4	6
D				0	3
EF					0

- E and F

2. Continue from the previous question, what is the distance between the new cluster and A?

- 7

dist(A, EF) = max{AF, AE} = max{7, 6} = 7

3. Suppose the algorithm stops when there are 3 clusters. Which one of the following is one of the three clusters?

- $\{A, B\}, \{C\}, \{D, E, F\}$

$\frac{8, 10, X}{3} = 5$ $\{-3, 5, X, 8, 10\}$

$\Rightarrow X = -3$

$C_1 = 5, 5, 7 \quad X$

$C_2 = 10, 8, 10$

$\frac{X+5+7}{3} = 5$

$\Rightarrow X = 3$

$\{3, 5, 7, 8, 10\}$

4. Suppose K-Means with $K = 2$ is used to cluster the dataset $\{5, 7, 8, 10, x\}$ and initial cluster centers $c_1 = 5, c_2 = 10$. What is x if one of the cluster centers in the next iteration is 5? Here, x can belong to either cluster 1 or 2.

- 3

C_2

5. Continue from the previous question (but ignore the last sentence). What is x if one of the cluster centers in the next iteration is 7? Here, x can belong to either cluster 1 or 2.

$\{5, 7, 8, 9, 10\}$

- Impossible (9 would belong to the other cluster)

$\frac{5+7+X}{3} = 7 \Rightarrow X = 9$

$\frac{8+10+X}{3} = 7 \Rightarrow X = 3$

$\{3, 5, 7, 8, 10\}$

6. What is the projection of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

- $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

7. The projection of $\begin{bmatrix} u \\ v \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length 2. What is the projection of $\begin{bmatrix} u \\ v \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

- $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$x_1 \quad x_2 \quad u$

8. What is the projected sample variance of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Note: projected sample variance is the sample variance of the magnitude of the projection of the data points onto a principal component, which $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in this question. Use the maximum likelihood estimator of σ^2:

$(1,3)$ length=3

$$\frac{x_1^T u}{u^T u} u = \frac{(1,2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{x_2^T u}{u^T u} u = \frac{(3,4)\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$(1,3)$ $\hat{\mu}=2$

$$(\hat{\sigma})^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2 \quad \boxed{\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

- 1

length=1 $\sqrt{1^2+0^2} = 1$

$\hat{u} = \frac{1+3}{2} = 2$

$\sigma^2 = \frac{1}{2}\left((1-2)^2 + (3-2)^2\right) = 1$

9. Continue from the previous question. What is the projected sample variance of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$x_1 \quad x_2 \quad u$

$$\frac{x_1^T u}{u^T u} u = \frac{(1,2)\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(0,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$ length=2

- 1

$$\frac{x_2^T u}{u^T u} u = \frac{(3,4)\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(0,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$ length=4

$\hat{\mu} = \frac{2+4}{2} = 3 \quad \sigma^2 = \frac{1}{2}\left((2-3)^2 + (4-3)^2\right) = 1$

10. Suppose X^TX is a 100×100 matrix. Which one of the following expressions represents the entry of X^TX on row 30 column 60? Notation: e_i is the length 100 vector $(0, 0, ..., 0, 1, 0, ..., 0, 0)^T$ with a 1 at position i and 0 everywhere else, and e is the length 100 vector $(1, 1, ..., 1, 1)^T$ with 1 at every position.

- $e_{30}^T X^T X e_{60}$ or $e_{60}^T X^T X e_{30}$ (This is because X^TX is symmetric)

11. Continue from the previous question. Which one of the following expressions represents the sum of the entries of X^TX on row 60?

- $e_{60}^T X^T X e$ or $e^T X^T X e_{60}$

12. Continue from the previous question. Which one of the following expressions represents the sum of the entries of X^TX on column 30?

- $e_{30}^T X^T X e$ or $e^T X^T X e_{30}$

13. Suppose a face image is represented by 3 features X $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and the principal components (eigenfaces) are the following. What is the reconstructed face image using the one principal component ($K = 1$)?

$X = u_1^T x \, u_1 + u_2^T x \, u_2 + u_3^T x \, u_3$

$K=3$ new feature

$(u_1^T x_i, u_2^T x_i, \ldots u_k^T x_i)^T$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ with } \lambda_1 = 3 \quad \text{PC1}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ with } \lambda_2 = 2 \quad \text{PC2}$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ with } \lambda_3 = 1 \quad \text{PC3}$$

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} -$

- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (1, 2, 3)$$

14. Continue from the previous question. What is the reconstructed face image using two (the first two) principal components ($K = 2$)?

- $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$i=0$ $(i+2)\%7$

15. Suppose the states are integers between 0 and 6. The initial state is 0, and the goal state is 6. The successors of a state i are $i+2 \bmod 7$ and $i+5 \bmod 7$, if exist. What is a state expansion sequence if Breadth First Search (BFS) is used? Use the convention that a smaller integer is always enQueued before a larger integer, and a list of visited states are stored so that the same state is never enQueued twice. For example, enQueueing $\{3,5,7\}$ into the Queue with $\{1,7,9\}$ (from front to back) results in $\{1,7,9,3,5\}$ (from front to back). Use this convention for all search questions.

- 0, 2, 5, 4, 3, 6

0 2 5 4 3 6 1

$i=2$ $i=5$ $i=4$ $i=3$

16. Continue from the previous question. What is a state expansion sequence if Depth First Search (DFS) is used? Use the convention that a smaller integer is always pushed in a Stack after a larger integer, and a list of visited states are stored so that the same state is never pushed twice. For example, pushing $\{3,5,7\}$ into the Stack with $\{1,7,9\}$ (from top to bottom) results in $\{3,5,1,7,9\}$ (from top to bottom). Use this convention for all search questions.

- 0, 2, 4, 6

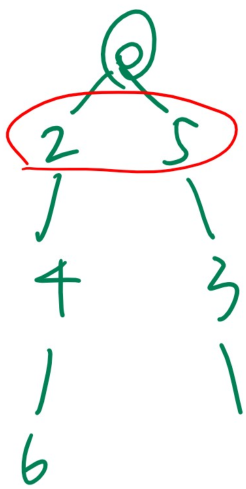
$(i+2)\%7$

$(i+5)\%7$

$i=2$

$(i+2)\%7=4$

$(i+5)\%7=0$



$i=5$ $(i+2)\%7=0$

$(i+5)\%7=3$

$i=4$

$(i+2)\%7=6$

$(i+5)\%7=2$

$i=3$

$(i+2)\%7=5$

$(i+5)\%7=1$

Q16

~~6~~

~~4~~

~~2~~

5

~~0~~

17. Suppose the states are integers between 1 and $2^{10} = 1024$. The initial state is 1, and the goal state is 1024. The successors of a state i are $2i$ and $2i + 1$, if exist. How many states are expanded during a BFS search?

- 1024

18. Continue from the previous question. How many states are expanded during a DFS search?

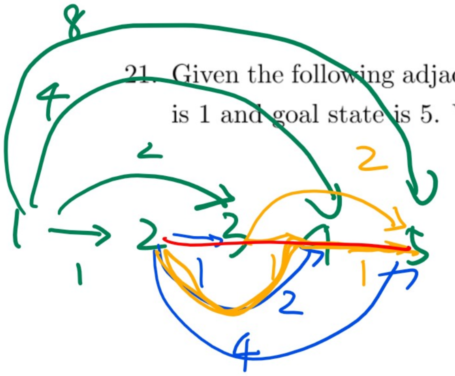
- 11

19. Suppose the states are integers between 1 and $2^{10} - 1 = 1023$. The initial state is 1, and the goal state is 1023. The successors of a state i are $2i$ and $2i + 1$, if exist. How many states are expanded during a BFS search?

- 1023

20. Continue from the previous question. How many states are expanded during a DFS search?

- 1023



21. Given the following adjacency matrix representing the edge costs of the state digraph. The initial state is 1 and goal state is 5. What is a vertex expansion sequence if Uniform Cost Search (UCS) is used?

init ↓ goal ↓

–	1	2	3	4	5
1	0	1	2	4	8
2	–	0	1	2	4
3	–	–	0	1	2
4	–	–	–	0	1
5	–	–	–	–	0

g min

- 1, 2, 3, 4, 5

	1	2	3	4	5
g	0	1	2	3	4

22. Continue from the previous question. What is a vertex expansion sequence if Best First Greedy Search is used?

h(s) min $h^*(s)$

$h(1) = 4, h(2) = 3, h(3) = 2, h(4) = 1, h(5) = 0$

- 1, 5

state	1	5
h	4	0

state	1	2	3	4	5
$h^*(s)$ true cost	4	3	2	1	0

h(2)

1 → 2 → 5

g h

state	1	2	3	4	5
$h^*(s)$	4	3	2	1	0

$\sqrt{x} < x, \; x > 1$

$\log_2 16 = 4$

$x = 5 - i$

5 - 1   5 - 2

23. Continue from the previous question. How many of the following heuristic functions are admissible? For $i = 1, 2, 3, 4, 5$:

$0 \le h(s) \le h^*(s)$

true cost

$h(s)$	1	2	3	4	5
$h(s) = 5 - i$	4	3	2	1	0
$\sqrt{5-i}$	$\sqrt{4}$	$\sqrt{3}$	$\sqrt{2}$	$\sqrt{1}$	0
$\log_2(6-i)$	$\log_2 5$	$\log_2 4$	$\log_2 3$	$\log_2 2$	$\log_2 1$
$1 - \mathbb{1}_{i=5}$	1	1	1	1	0

$h(s)$

- $h(i) = 5 - i$ ✓
- $h(i) = \sqrt{5 - i}$ ✓
- $h(i) = \log_2(6 - i)$ ✓
- $h(i) = 1 - \mathbb{1}_{i=5}$ ✓
- $h(i) = 0$ ✓

$\begin{cases} 1 & \text{otherwise} \\ 0, & i = 5 \end{cases}$

• 5

24. Continue from the previous question. Which one of the following heuristic functions is not dominated (among the admissible ones)? For $i = 1, 2, 3, 4, 5$

$h_1(s) \le h_2(s) \le h^*(s)$

h_2 dominates h_1

h_1 is dominated

- $h(i) = 5 - i = h^*(s)$ ✓
- $h(i) = \sqrt{5 - i}$
- $h(i) = \log_2(6 - i)$
- $h(i) = 1 - \mathbb{1}_{i=5}$
- $h(i) = 0$

• The first one.

25. Given the following tables showing the cell indices and the list of successors of a maze. The first table contains the names (indices) of the states for each cell, and the second table contains the indices of the successor states for each cells. The entrance is 1, the exit is 16. How many cells are visited if IDS is used? Do not count the same cell twice during the search. Count both initial and goal states.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

2	1, 3	2, 4	3, 8
6	5, 7	6, 8	4, 7, 12
10	9, 11	10, 12	8, 11, 16
14	13, 15	14, 16	12, 15

- 10

26. Continue from the previous question. How many cells are visited if UCS is used?

- 10

UCS + C=1 ⇒ BFS

27. Continue from the previous question. How many cells are visited if $A^\star$ is used with Manhattan distance between the cell and the goal?

- 7

min(g + h)

=3

h(7) → 16

g(8) <

h(12) → 16

1->8

=1

g(12) < h(11)

h(16) = 0

1->12

28. Starting from $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ (all variables set to False), what is the minimum number of steps required to reach the solution for the following SAT problem if first choice hill climbing is used? The neighbors differ by the value of one variable. The score is the number of clauses satisfied. The following express is 5 clauses connected by $\wedge$.

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_2 \vee x_3 \vee x_4)$$

- 2

29. Suppose the states are given by five integers from $\{0, 1\}$. The fitness function is the position of the first 1 in the sequence. $F(x_1, x_2, x_3, x_4, x_5) = \min\{t \in \{1, 2, 3, 4, 5, 6\} : x_t = 1\}$, with $x_6 = 1$. There are in total four states. Using the genetic algorithm, what is the reproduction probability of the first state a? The reproduction probabilities are proportional to the fitness of the states.

$$p_i = \frac{F(s_i)}{\sum F(s_i)}$$

$$x_a = (0, 0, 1, 0, 0) \quad F(x_a) = 3$$
$$x_b = (0, 1, 0, 0, 1) \quad F(x_b) = 2$$
$$x_c = (0, 0, 1, 1, 0) \quad F(x_c) = 3$$
$$x_d = (0, 0, 0, 0, 0) \quad F(x_d) = 6$$

- $\frac{3}{14}$

$$P_a = \frac{F(x_c)}{F(x_a)+F(x_b)+F(x_c)+F(x_d)} = \frac{3}{3+2+3+6} = \frac{3}{14}$$

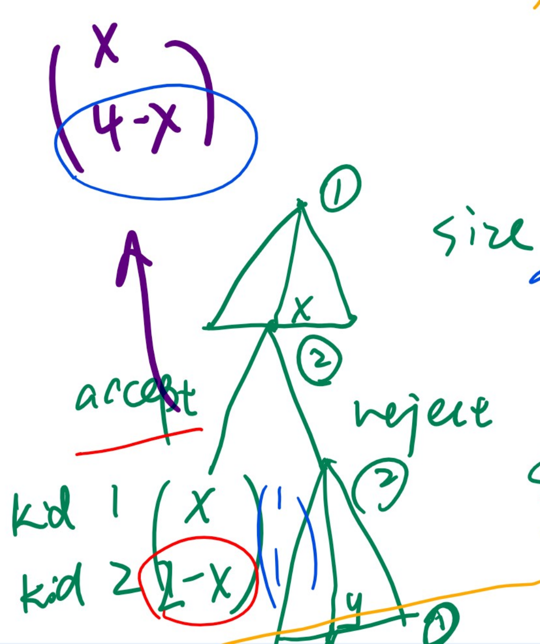
30. Continue from the previous question. Suppose one new state is added, x_e, and the reproduction probability of x_d is $\frac{1}{3}$. What is the fitness of the new state, e?

- 4

$$P_d = \frac{F(x_d)}{F(x_a)+F(x_b)+F(x_c)+F(x_d)+F(x_e)} = \frac{1}{3}$$

$$\frac{6}{3+2+3+6+F(x_e)} = \frac{1}{3}$$

$$F(x_e) = 4$$



size = 2 θ31
size = 4 θ32

kid 2 accept iff

$2 - x \geq 1 - y = 1 \Rightarrow x \leq 1$

θ31
size = 1
size = 3 θ32

$x = 1$

$x = 2$

$4 - x \geq 3 - y = 2 \Rightarrow x \leq 2$

accept reject

Kid 1: Accept if and only if $y \geq 0$ $y = 0$

$y \geq 1$ $y = 1$

Kid 1

Kid 2

31. Imagine two kids arguing about how to divide a cake with T pieces and dad steals 1 piece every time they cannot reach agreement. Let $t = T$ at the beginning of a two player game. In the first round, player 1 propose an integer division (partition) of t, say integers (a, b) with $a + b = t$, and player 2 decides whether to accept the division. If player 2 accepts the division the game is over, player 1 gets a, and player 2 gets b. In the second round, $t = T - 1$, player 2 propose an integer division of t, say integers (c, d) with $c + d = t$, and player 1 decides whether to accept the division. The process is repeated for $t = T - 2$, player 1 propose, $t = T - 3$, player 2 propose, and so on until $t = 0$. When $t = 0$, both players get 0. Assume if a player is indifferent, the player will accept the division. A solution (also called subgame perfect equilibrium) is the path from the beginning of the game to the one ending of the game under the assumption that players choose the optimal action (successor) at all possible states on the game. When $T = 2$, in the solution of the game, what is the proposal player 1 makes in the first round?

- $(1, 1)$

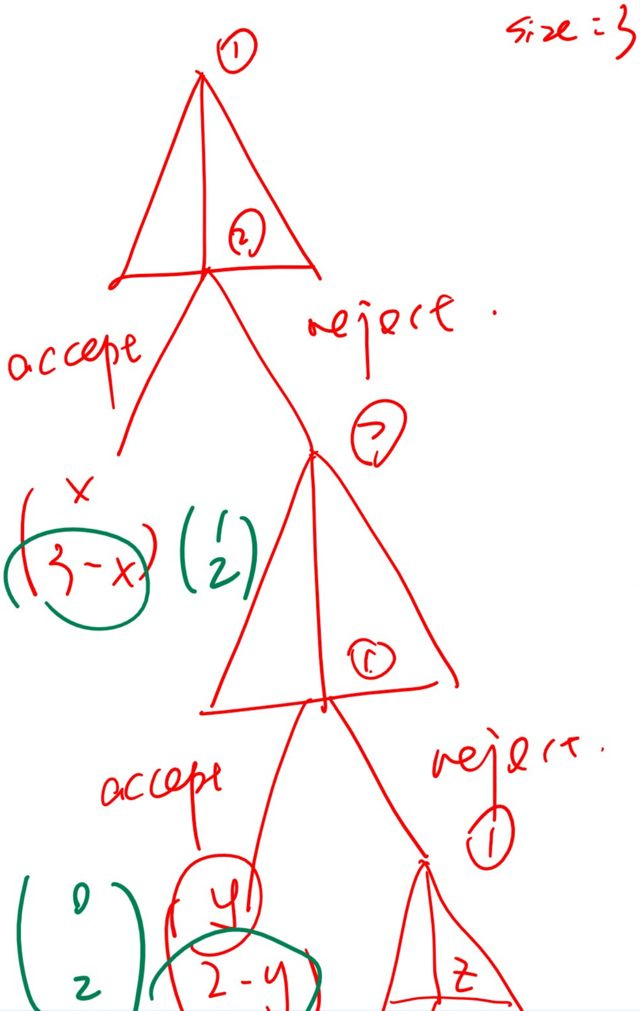
32. Continue from the previous question. The rule and the solution concept is the same but with $T = 4$. In the solution of the game, what is proposal player 1 makes in the first round?

- $(2, 2)$

33. Continue from the previous question. The rule and the solution concept is the same but with $T = 1024$. In the solution of the game, what is proposal player 1 makes in the first round?

- $(512, 512)$

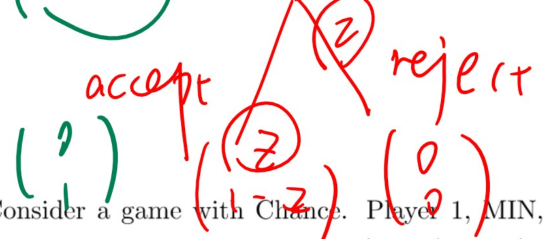
size = 3



② $3 - x \geq 2 - y = 2$ $x = 1$

① $y \geq z$ $y = 0 = z$

② $z \geq 0$. $z = 0$



$z \geq 0$ $z = 0$

34. Consider a game with Chance. Player 1, MIN, chooses an action H or T first. Then a fair coin is flipped, the outcome is either H (heads) or T (tails). Player 2, MAX, observes the outcome of the coin and chooses an action H or T. Player 2 wins if all three actions are the same. What is the value of the (whole) game? The value is 1 if player 2 wins and -1 otherwise.

- 0

35. Consider a zero-sum sequential move game in which player MAX moves first, and MIN moves next. Each player has three actions labeled $1, 2, 3$. The value to the MAX player if MAX plays $x_1 \in \{1, 2, 3\}$ and MIN plays $x_2 \in \{1, 2, 3\}$ is $(4 - x_1)(4 - x_2)$. In Alpha Beta pruning is used. What is the number of branches (states) that can be pruned? During the search process, the actions with smaller labels are searched first. Note that a branch is pruned if $\alpha = \beta$.

- 2

36. For a zero-sum simultaneous move game, the ROW player is the MAX player and the COL player is the MIN player. Given the following game payoff table, how many states survive the iterated elimination of weakly dominated strategies?

MIN

$0 \geq 0,\ 1 \geq 0,\ 2 \geq 0 \Rightarrow U \geq M$

col : MIN (0, 1, 2)

MAX

–	L	M	R
U	0	1	2
M	0	0	0
D	0	-1	-2

row : MAX

$U \geq M \geq D$

row col

row

- 1 (U, L)

col

37. Continue from the previous question, how many states survive the iterated elimination of strictly dominated strategies?

- 9

$U \geq M$

38. Continue from the previous question, which one of the following is a Nash equilibrium? (U, L) is not one of the choices.

column

- (M, L)

Row player

→ U L = 0 (0, 1, 2)

→ M L, M, R = 0 (0, 0, 0)

→ D R = -2 (0, -1, -2)

(U, L)

(M, L) (M, M) (M, R)

(D, R)

column player ↓L ↓M ↓R

(U, L) (U, M) (U, R)

(M, L)

(D, L)

U = 0 U = 1 U = 2

M = 0 M = 0

D = 0 D = -2

0 1

0 0

0 -1

39. Given the following game payoff table, what is the complete set of pure strategy Nash equilibria? Both players are MAX players. In each entry of the table, the first number is the cost to the ROW player and the second number is the cost to the COLUMN player.

row col MAX

–	L	R
U	A (0, 0)	B (1, 1)
D	C (1, 1)	D (1, 1)

MAX

$\frac{1}{2}$ $\frac{1}{2}$

row col

(U, R)

- B, C, D

(D, R)

(D, L)

40. Continue from the previous question, how many of the following are mixed strategy Nash equilibria of the game?

always R

fix row, check col

U $L = 0$, $R = 1$

D $L = 1$, $R = 1$

fix col, check row

$$\left(\text{always } U, \left(L\frac{1}{2} \text{ of the time}, \text{ R } \frac{1}{2} \text{ of the time}\right)\right)$$

$$\left(\text{always } D, \left(L\frac{1}{2} \text{ of the time}, \text{ R } \frac{1}{2} \text{ of the time}\right)\right)$$

always D

$$\left(\left(U\frac{1}{2} \text{ of the time}, \text{ D } \frac{1}{2} \text{ of the time}\right), \text{ always } L\right)$$

$$\left(\left(U\frac{1}{2} \text{ of the time}, \text{ D } \frac{1}{2} \text{ of the time}\right), \text{ always } R\right)$$

fix col, check row

$br_{row}(\frac{1}{2}L, \frac{1}{2}R)$

U: $0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

(U, L) (U, R)

D: $1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$

(D, L) (D, R)

- 2, second and fourth

fix row, check col

$br_{col}(\frac{1}{2}U, \frac{1}{2}D)$

L: $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

(U, L) (D, L)

R: $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$

(U, R) (D, R)

fix row, check col

(always D, any mix)

check row, fix col) $(\frac{1}{4}L, \frac{3}{4}R)$

$br_{row}(\frac{1}{4}L, \frac{3}{4}R) =$ U: $\frac{1}{4} \cdot ? + \frac{3}{4} \cdot ?$

D: $\frac{1}{4} \cdot ? + \frac{3}{4} \cdot ?$

41. Calculator?

- No

42. Number of pages of additional notes? Please submit them at the end of the exam.

- 0

43. Bonus mark: You received one free point for this question and you have two choices: donate 1 and keep 0 or keep 1 and donate 0. Your final grade is the points you keep plus twice the average donation (sum of donation from everyone in your section divided by the number of people in your section).

- ?