CS540 Introduction to Artificial Intelligence
Lecture 20

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Based on lecture slides by Jerry Zhu, Yingyu Liang, and Charles Dyer

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Guess Average Game

Motivation

M12Q9?

- Write down an integer between 0 and 100 that is the closest to two thirds (2/3) of the average of everyone's (including yours) integers.

simultaneous move games.

IESDS

0 .. . - - 65 66 ~~67 100~~

impossible to be best response.

0 . . - - - - 44 ~~45 66~~

0 . . . - . 30 ~~31 44~~

[0 1] → rationalizable actions

Rationalizability

Motivation

- An action is 1-rationalizable if it is the best response to some action.
- An action is 2-rationalizable if it is the best response to some 1-rationalizable action.
- An action is 3-rationalizable if it is the best response to some 2-rationalizable action.
- An action is rationalizable if it is ∞-rationalizable.

Best Response

Definition

- An action is a best response if it is optimal for the player given the opponents' actions.

payoff / value

$$br_{MAX}(s_{MIN}) = \arg\max_{s \in S_{MAX}} c(s, s_{MIN})$$

$$br_{MIN}(s_{MAX}) = \arg\min_{s \in S_{MIN}} c(s_{MAX}, s)$$

min other players value

Strictly Dominated and Dominant Strategy

Definition

- An action s_i strictly dominates another $s_{i'}$ if it leads to a better state no matter what the opponents' actions are.

for all

$$s_i \succ_{MAX} s_{i'} \text{ if } c(s_i, s) > c(s_{i'}, s) \ \forall s \in S_{MIN}$$
$$s_i \succ_{MIN} s_{i'} \text{ if } c(s, s_i) < c(s, s_{i'}) \ \forall s \in S_{MAX}$$

- The action $s_{i'}$ is called strictly dominated.
- An action that strictly dominates all other actions is called strictly dominant.

Nash Equilibrium

Definition

- A Nash equilibrium is a state in which all actions are best responses.

Dominated Strategy Example 1

Quiz

- Fall 2005 Final Q6
- Both players are MAX players. What are the dominated strategies for the ROW player? Choose E if none of the strategies are dominated.

–	A	B	C
A	(2, 4)	(3, 7)	(4, 5)
B	(1, 2)	(5, 4)	(2, 3)
C	(4, 1)	(2, 8)	(5, 3)
D	(3, 6)	(4, 0)	(1, 9)

col

Row Col

(B, B) is the only pair of rationalizable actions.

IESDS

after A, C eliminated

B > A, C

Row

after D is eliminated

B > C

for col no matter what row is playing

A < C

∀ col, row D < B

Dominated Strategy Example 2

Quiz

- Fall 2005 Final Q6
- Both players are MAX players. What are the dominated strategies for the COLUMN player? Choose E if none of the strategies are dominated.

Nash Equilibrium

–	A	B	C
A	(2, 4)	(3, 7)	(4, 5)
B	(1, 2)	(5, 4)	(2, 3)
C	(4, 1)	(2, 8)	(5, 3)
D	(3, 6)	(4, 0)	(1, 9)

NE

(B, B)

mutual best response

br_{COL} br_{ROW}

$br_{COL}(D) = C$ $br_{ROW}(C) = C \neq \cancel{D}$ $\Rightarrow$ Nash equilibrium

Nash Equilibrium Example 1

Quiz

Q7

NE $\subseteq$ rationalizable actions

subset of

IESDS ←

- Find the value of the Nash equilibrium of the following zero-sum game.

MIN

MAX

–	I	II	III
I	-4	-7	-3
II	9	1	7
III	-6	-1	5

-7

1

-6

max min

9 1 7

min max

- A: -7 , B: 9 , C: -3 , D: 1, E: -4

Nash Equilibrium Example 2

Quiz

Q7

Col max col's value
or min row's value.

Row

- Find the value (of MAX player) of the Nash equilibrium of the following zero-sum game.

–	I	II	III
I	(-4, 4)	(-7, 7)	(-3, 3)
II	(9, −9)	(1, −1)	(7, −7)
III	(-6, 6)	(-1, 1)	(5, −5)

- A: -7 , B: 9 , C: -3 , D: 1, E: -4

proj from
I = II = III

pure
I, II, III
against mixed

p q 1-p-q → system of 2 eq

p 1-p 0
p p 0 1 eq.

Public Good Game

Discussion

- You received one free point for this question and you have two choices.
- A: Donate the point.
- B: Keep the point.
- Your final grade is the points you keep plus twice the average donation.

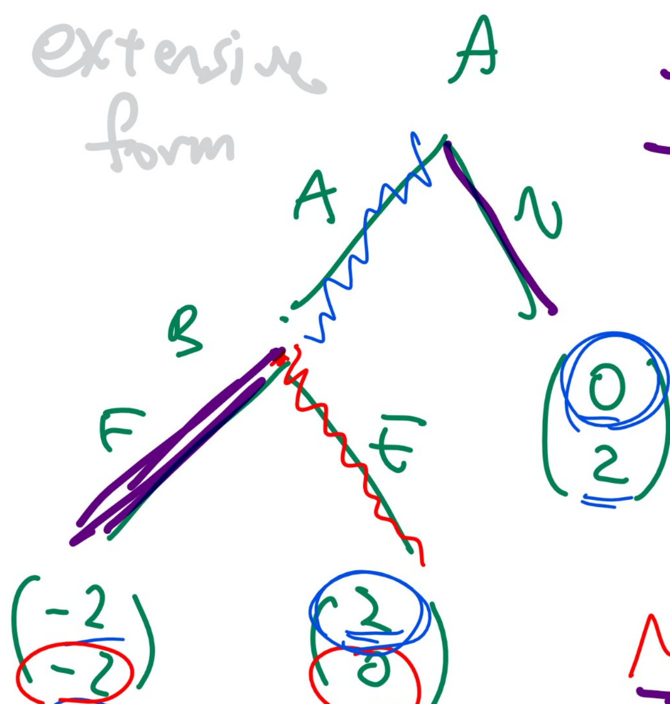
Non-credible Threat Example 1

Quiz

- Country A can choose to Attack or Not attack country B. If country A chooses to Attack, country B can choose to Fight back or Escape. The costs are the largest for both countries if they fight, but otherwise, A prefers attacking (and B escaping) and B prefers A not attacking. What are the Nash equilibria?
- A: (A, F)
- B: (A, E)
- C: (N, F)
- D: (N, E)
- E: (N)

Solution of game = (A, E)

extensive form



normal form

	F	E
A	(-2, -2)	(2, 0)
N	(0, 2)	(0, 2)

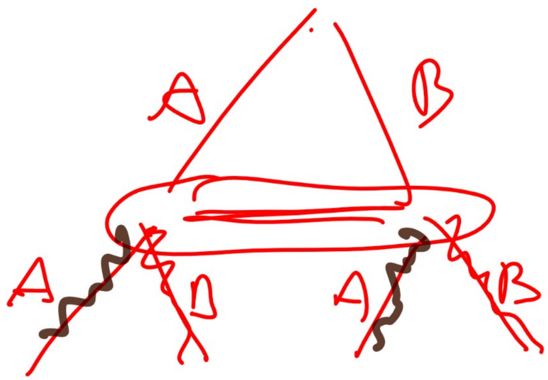
NE: (A, E) (N, F)

Non-credible Threat Example 1 Derivation

Quiz

extensive form

Simultaneous move



normal form

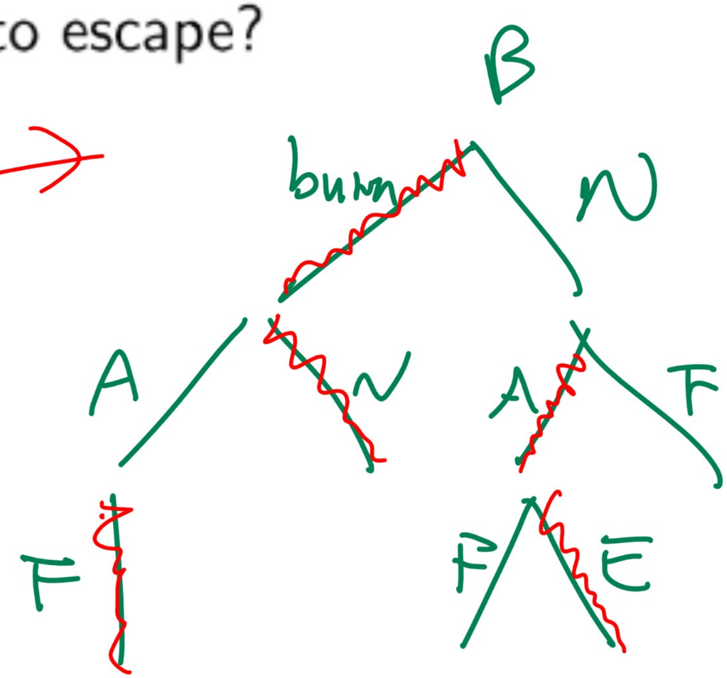
	A	B
A		
B		

Non-credible Threat Example 2

Quiz

- What if country B can burn the bridge at the beginning of the game so that it cannot choose to escape?

credible threat



solution
NE

Mixed Strategy Nash Equilibrium

Definition

- A mixed strategy is a strategy in which a player randomizes between multiple actions.
- A pure strategy is a strategy in which all actions are played with probabilities either 0 or 1.
- A mixed strategy Nash equilibrium is a Nash equilibrium for the game in which mixed strategies are allowed.

Battle of the Sexes Example

Discussion

- Battle of the Sexes (BoS, also called Bach or Stravinsky) is a game that models coordination in which two players have different preferences in which alternative to coordinate on.

Juliet

—	Bach	Stravinsky
Bach	A (x, y)	$B\,(0,0)$
Stravinsky	$C\,(0,0)$	D (y, x)

Romeo

Battle of the Sexes Example 1

Quiz

- Find all Nash equilibria of the following game.

—	I	II
I	$\boxed{A\,(3,5)}$	$B\,(0,0)$
II	$C\,(0,0)$	$\boxed{D\,(5,3)}$

✓ pure strategy, NE

Battle of the Sexes Example 1 Derivation 1

Quiz

for row player

if B payoff is $3 \cdot q + 0(1-q)$

if S payoff is $0 \cdot q + 5(1-q)$

- Find all mixed strategy Nash equilibria of the following game.

		q	$1-q$
	–	B	~~H~~ S
p	B	$(3,5)$	$(0,0)$
$1-p$	~~H~~ S	$(0,0)$	$(5,3)$

$$br_{row}(\underset{col}{q}) = \begin{cases} B \\ p \in [0,1] \quad B, S \\ S \end{cases}$$

$3q \geq 5(1-q)$

$q \geq \frac{5}{8}$

$\boxed{q = \frac{5}{8}}$

$q \leq \frac{5}{8}$

Battle of the Sexes Example 1 Derivation 2

Quiz

$$br_{col}(p) = \begin{cases} B \\ q \in [0,1] \\ S \end{cases}$$

$5p \geq 3(1-p)$

$p \geq \frac{3}{8}$

$\boxed{p = \frac{3}{8}}$

$p \leq \frac{3}{8}$

		q	$1-q$
	—	~~H~~ B	~~H~~ S
p	B	(3,5)	(0,0)
$1-p$	~~H~~ S	(0,0)	(5,3)

for col

payoff from B

$5 \cdot p + 0(1-p)$

payoff from S

$0 \cdot p + 3(1-p)$

mixed NE = $\left(B^{(\frac{3}{8})} S^{(\frac{5}{8})}, B^{(\frac{5}{8})} S^{(\frac{3}{8})}\right)$

Mixed Strategy Example 1

Quiz

- Which ones of the following are Nash equilibria?

Q p (last) for Row

If $U \Rightarrow \boxed{1.5}$

$D \Rightarrow 0.5$

—	L	R
U	(3, 1)	(0, 0)
D	(0, 1)	(1, 1)

$br_{COL}(U)$ = always L.

- ~~A~~: (always U , ~~(L $\frac{1}{2}$ of the time , R $\frac{1}{2}$ of the time)~~)

not br

$br_{Row}(\frac{1}{2},\frac{1}{2}) = U \,\&\, D$

- B: (~~always D~~ , (L $\frac{1}{2}$ of the time , R $\frac{1}{2}$ of the time))

U is br

- C: ((~~U $\frac{1}{2}$ of the time , D $\frac{1}{2}$ of the time~~) , always L)

$br(L) = U$

- D: ((U $\frac{1}{2}$ of the ~~time~~ , D $\frac{1}{2}$ of the time) , always R)

$br(R) = D$

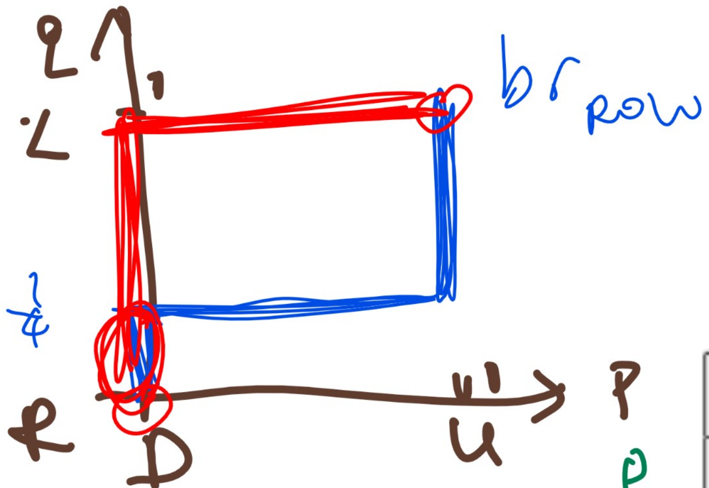
- E: ((~~U $\frac{1}{2}$ time , D $\frac{1}{2}$ time~~) , (L $\frac{1}{2}$ time , R $\frac{1}{2}$ time))

$br(\frac{1}{2},\frac{1}{2}) = U$

Mixed Strategy Example 1 Derivation

Quiz

post a video
also UCS



COL

		q	$1-q$
	—	L	R
p	U	(3,1)	(0,0)
$1-p$	D	(0,1)	(1,1)

L R

$3q + 0(1-q) \geq 0q + 1(1-q)$

$\Downarrow$

$$br_{COL}(p) = \begin{cases} L & 1 \geq (1-p)1, \quad p \geq 0 \\ q \in [0,1] & \boxed{p = 0} \\ R & p \leq 0 \end{cases}$$

$$br_{ROW}(q) = \begin{cases} U & q \geq \frac{1}{4} \\ p \in [0,1] & q = \frac{1}{4} \\ D & \boxed{q \leq \frac{1}{4}} \end{cases}$$

Nash Theorem

Definition

$$\begin{cases} NE = \left(D,\ L^{q \le \frac{1}{4}} R^{1-q}\right) \\ \text{e.g. } NE = \left(D,\ L^{(\frac{1}{4})} R^{(\frac{3}{4})}\right),\ \left(D,\ L^{(\frac{1}{8})} R^{(\frac{7}{8})}\right) \end{cases}$$

- Every finite game has a Nash equilibrium. $(D, R) \rightarrow C^0 R^1$
- The Nash equilibria are fixed points of the best response functions.

P4 :

$$\frac{x_i - \min}{\max - \min} \text{ in dim } i$$

$$\frac{x - 0}{255 - 0}$$

Fixed Point Nash Equilibrium

Algorithm

- Input: the payoff table $c(s_i, s_j)$ for $s_i \in S_{MAX}, s_j \in S_{MIN}$.
- Output: the Nash equilibria.
- Start with random state $s = (s_{MAX}, s_{MIN})$.
- Update the state by computing the best response of one of the players.

$$\text{either } s' = (br_{MAX}(s_{MIN}), br_{MIN}(br_{MAX}(s_{MIN})))$$
$$\text{or } s' = (br_{MAX}(br_{MIN}(s_{MAX})), br_{MIN}(s_{MAX}))$$

- Stop when $s' = s$.