CS540 Introduction to Artificial Intelligence
Lecture 2

Young Wu
Based on lecture slides by Jerry Zhu and Yingyu Liang

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Quiz (Participation)

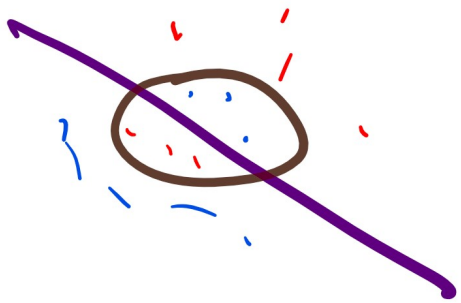
Guess Real Face

- Which one is the real face?
- A: Left
- B: Right
- C: Don't choose this
- D: Don't choose this
- E: Don't choose this

Activation Function

Review

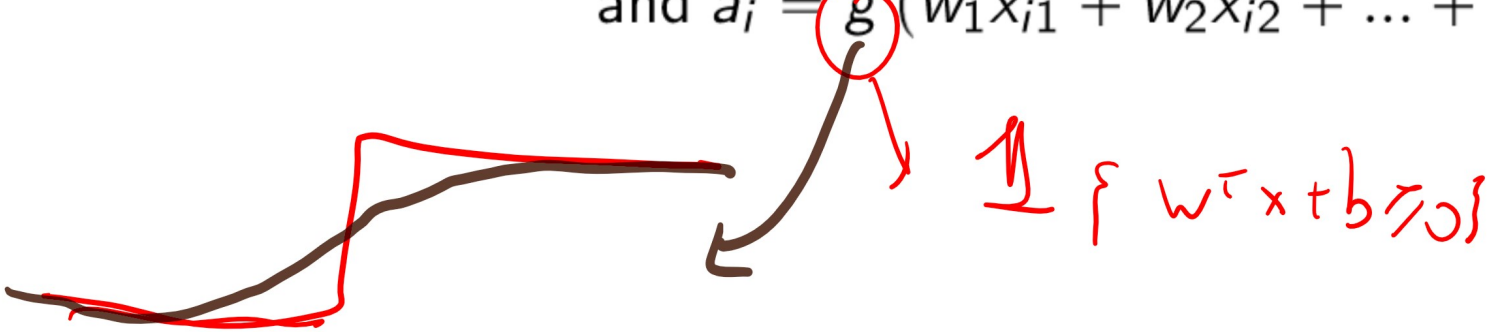
- The supervisered learning problem with activation function is the following.



$$\left(\hat{w}_0, \hat{w}_1, ..., \hat{w}_m, \hat{b}\right) = \arg \min_{w_1,...,w_m,b} C$$

$$\text{where } C = \frac{1}{2}\sum_{i=1}^{n}(a_i - y_i)^2$$

$$\text{and } a_i = g(w_1 x_{i1} + w_2 x_{i2} + ... + w_m x_{im} + b)$$



Sigmoid Activation Function

Motivation

- When the activation function g is the sigmoid function, the problem is called logistic regression.

$$g\left(\boxed{\cdot}\right) = \frac{1}{1 + \exp\left(-\boxed{\cdot}\right)}$$

- This g is also called the logistic function.

Sigmoid Function Diagram

Motivation

Cross Entropy Loss Function

Motivation

$C = \frac{1}{2}\sum (y - a_i)^2$

- The cost function used for logistic regression is usually the log cost function.

$$C = -\sum_{i=1}^{n} \left(y_i \log \left(f\left(x_i\right)\right) + \left(1 - y_i\right) \log \left(1 - f\left(x_i\right)\right)\right)$$

a_i a_i

- It is also called the cross-entropy loss function.

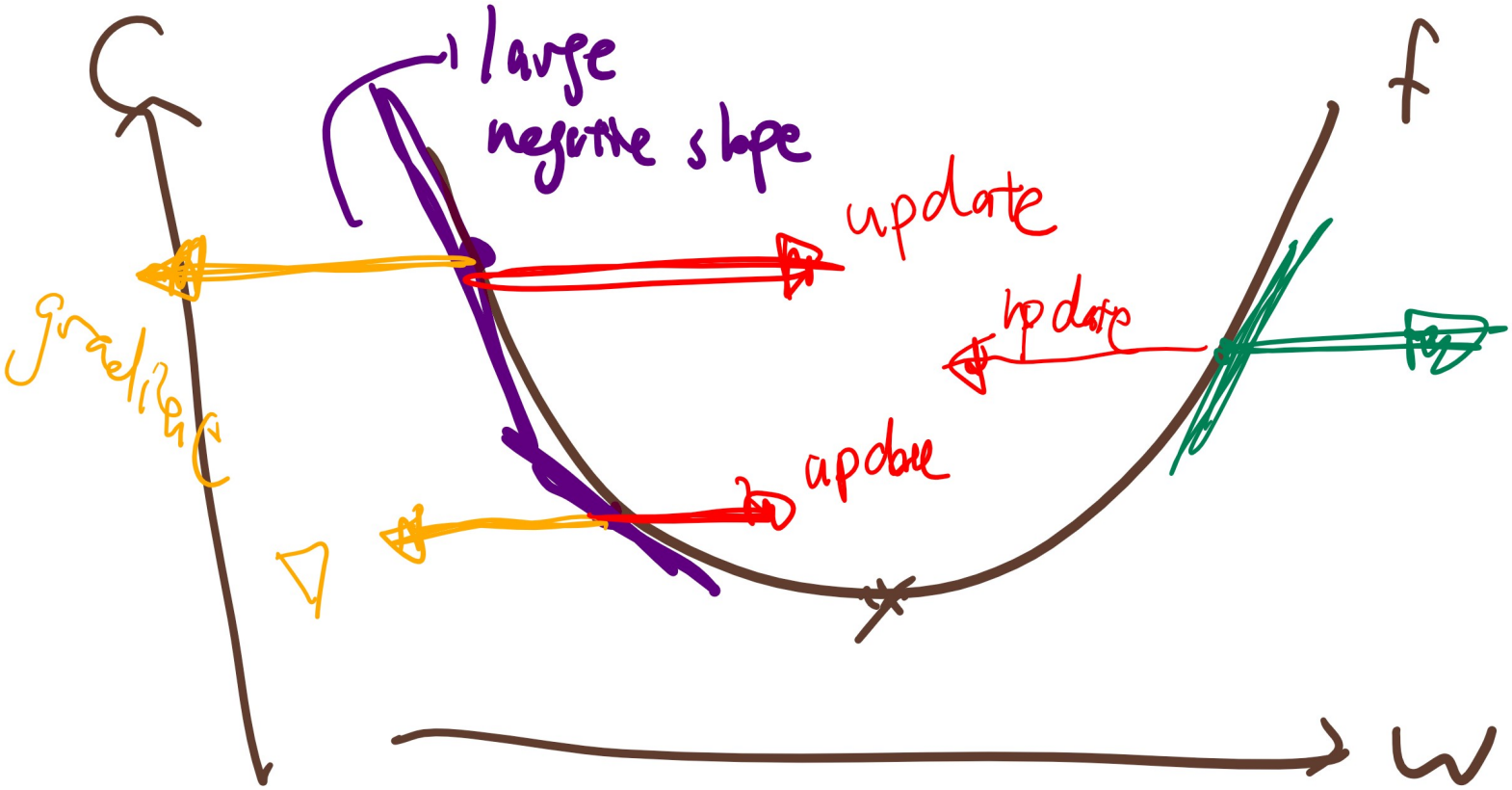
Logistic Regression

Description

- Initialize random weights.
- Evaluate the activation function.
- Compute the gradient of the cost function with respect to each weight and bias.
- Update the weights and biases using gradient descent.
- Repeat until convergent.

Optimization Diagram

Description



Gradient Descent Intuition

Definition

- If a small increase in w_1 causes the distances from the points to the regression line to decrease: increase w_1.
- If a small increase in w_1 causes the distances from the points to the regression line to increase: decrease w_1.
- The change in distance due to change in w_1 is the derivative.
- The change in distance due to change in $\begin{bmatrix} w \\ b \end{bmatrix}$ is the gradient.

Gradient

Definition

- The gradient is the vector of derivatives.
- The gradient of

$$f(x_i) = w^T x_i + b = w_1 x_{i1} + w_2 x_{i2} + \ldots + w_m x_{im} + b$$ is:

$$\nabla_w f = \begin{bmatrix} \frac{\partial f}{\partial w_1} \\ \frac{\partial f}{\partial w_2} \\ \ldots \\ \frac{\partial f}{\partial w_m} \end{bmatrix} = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \ldots \\ x_{im} \end{bmatrix} = x_i$$

$$\nabla_b f = 1$$

Chain Rule

Definition

- The gradient of
  $f(x_i) = g\left(w^T x_i + b\right) = g\left(w_1 x_{i1} + w_2 x_{i1} + \ldots + w_m x_{im} + b\right)$
  can be found using the chain rule.

$$\nabla_w f = g'\left(w^T x_i + b\right) x_i$$

$$\nabla_b f = g'\left(w^T x_i + b\right)$$

$f(x) = g(h(x))$

$\rightarrow g'(h(x)) \cdot \nabla h(x)$

- In particular, for the logistic function g :

$$g\left(\boxed{\cdot}\right) = \frac{1}{1 + \exp\left(-\boxed{\cdot}\right)}$$

$$g'\left(\boxed{\cdot}\right) = g\left(\boxed{\cdot}\right)\left(1 - g\left(\boxed{\cdot}\right)\right)$$

Logistic Gradient Derivation

Definition

$$g(x) = \frac{1}{1+e^{-x}} = (1+e^{-x})^{-1}$$

$$g'(x) = (1+e^{-x})^{-2} \cdot (-1) \cdot (e^{-x}) \cdot (-1)$$

$$= \frac{e^{-x}}{1+e^{-x}} \cdot \frac{1}{1+e^{-x}} = \left(1 - \frac{1}{1+e^{-x}}\right) \cdot \frac{1}{1+e^{-x}}$$

$$= (1-g(x))\, g(x)$$

Gradient Descent Step

Definition

- For logistic regression, use chain rule twice.

$$w = w - \alpha \sum_{i=1}^{n} (a_i - y_i)\, x_i$$

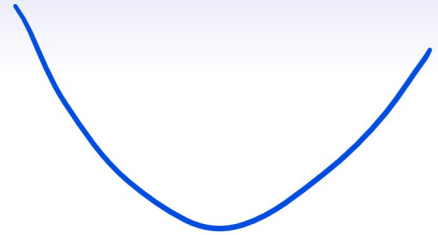
$$b = b - \alpha \sum_{i=1}^{n} (a_i - y_i)$$

$$a_i = g\left(w^T x_i + b\right), g\left(\boxed{\cdot}\right) = \frac{1}{1 + \exp\left(-\boxed{\cdot}\right)}$$

- α is the learning rate. It is the step size for each step of gradient descent.

Gradient Descent Derivation

Definition



$$C = -\sum_{i=1}^{n} y_i \log a_i + (1-y_i) \log(1-a_i)$$

$$a_i = g(w^T x + b)$$

$$\nabla_w C: \quad \frac{\partial C}{\partial w_j} = \sum_{i=1}^{n} \frac{\partial C}{\partial a_i} \cdot \frac{\partial a_i}{\partial w_j}$$

by Chain Rule

$$\frac{\partial C}{\partial a_i} = -\frac{y_i}{a_i} + \frac{1-y_i}{1-a_i}(-1)$$

$$\frac{\partial a_i}{\partial w_j}: \quad a_i \quad 1-a_i$$

$$g(w^T x+b) \cdot (1-g(w^T x+b)) \cdot x_j$$

g' $\quad \nabla_w (w^T x + b)$

Learning Rate Diagram

Definition

$$\sum_i \left(-\frac{y_i}{q_i} + \frac{1-y_i}{1-q_i} \right) q_i (1-q_i) \cdot x_i$$

$$\sum_i \left(-y_i(1-q_i) + q_i(1-y_i) \right) x_i$$

$$\sum_i (q_i - y_i) x_i$$

Gradient Descent

Quiz (Graded)

- What is the gradient descent step for w if the objective (cost) function is the squared error?

$$C = \frac{1}{2}\sum_{i=1}^{n}(a_i - y_i)^2$$

$$\nabla_w a_i = a_i(1-a_i)x_i$$

$$\nabla_w C = \sum_{i=1}^{n} \frac{\partial C}{\partial a_i} \nabla_w a_i$$

$$w = w - \alpha \nabla_w C$$

- A: $w = w - \alpha \sum (a_i - y_i)\, x_i$
- B: $w = w - \alpha \sum (a_i - y_i)\, a_i x_i$
- C: $w = w - \alpha \sum (a_i - y_i)\, (1 - a_i)\, x_i$
- D: $w = w - \alpha \sum (a_i - y_i)\, a_i\, (1 - a_i)\, x_i$
- E: None of the above

Gradient Descent, Another One

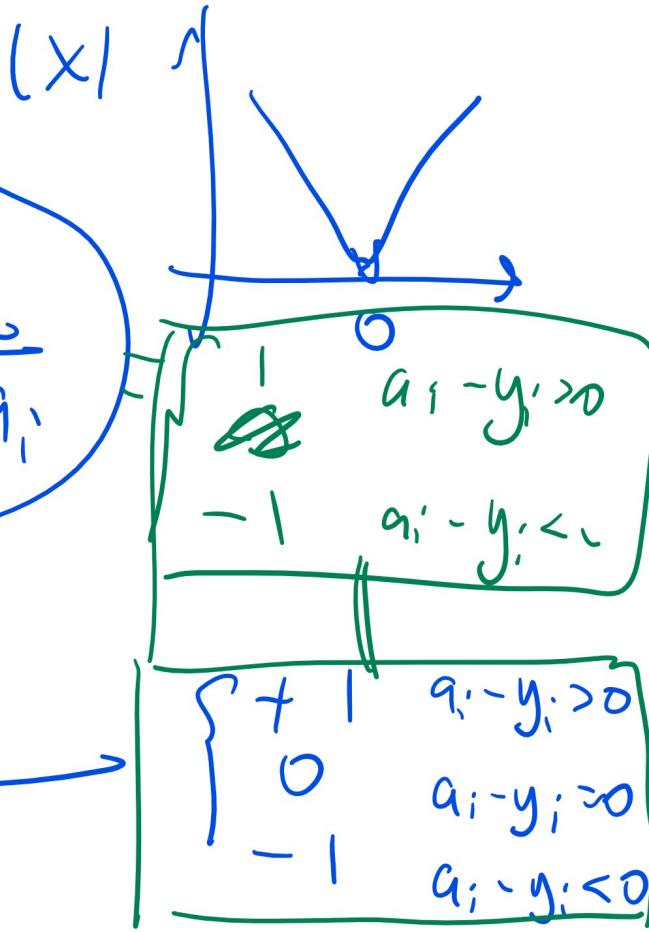
Quiz (Graded)

- What is the gradient descent step for w if the objective (cost) function is the absolute value error?

$$|a_i - y_i| = \begin{cases} a_i - y_i & \text{if } a_i - y_i \geq 0 \\ -(a_i - y_i) & \text{if } a_i - y_i < 0 \end{cases}$$

$$\sum_{i=1}^{n} |a_i - y_i|$$

$$\frac{\partial C}{\partial a_i}$$



- A: $w = w - \alpha \sum (a_i - y_i)\, a_i (1 - a_i)\, x_i$
- B: $w = w - \alpha \sum |a_i - y_i|\, a_i (1 - a_i)\, x_i$
- C: $w = w - \alpha \sum \mathbb{1}_{\{a_i - y_i > 0\}}\, a_i (1 - a_i)\, x_i$
- D: $w = w - \alpha \sum \text{sign}(a_i - y_i)\, a_i (1 - a_i)\, x_i$
- E: None of the above

Logistic Regression, Part 1

Algorithm

- Inputs: instances: $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$
- Outputs: weights and biases: $w_1, w_2, ..., w_m$ and b
- Initialize the weights.

$$w_1, ..., w_m, b \sim \text{ Unif } [0, 1]$$

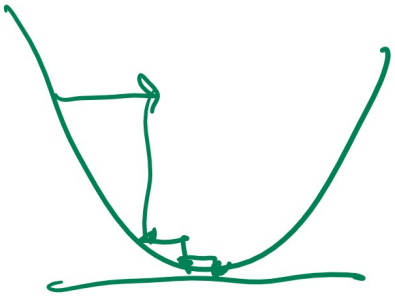
- Evaluate the activation function.

$$a_i = g\left(w^T x_i\right), g\left(\boxed{\cdot}\right) = \frac{1}{1 + \exp\left(-\boxed{\cdot}\right)}$$

Logistic Regression, Part 2

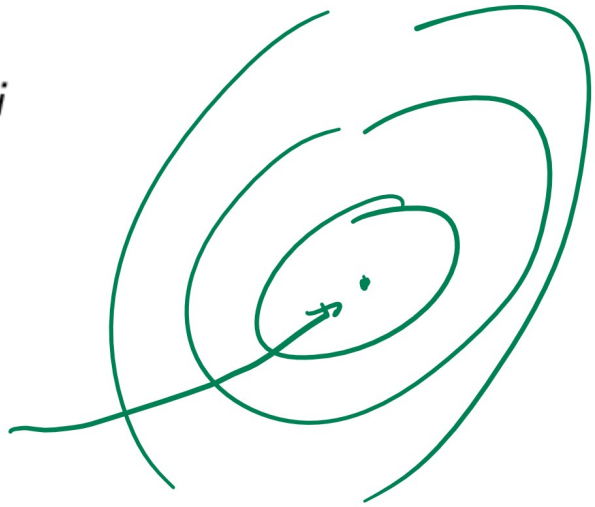
Algorithm

- Update the weights and bias using gradient descent.



$$w = w - \alpha \sum_{i=1}^{n} (a_i - y_i)\, x_i$$

$$b = b - \alpha \sum_{i=1}^{n} (a_i - y_i)$$



- Repeat the process until convergent.

$$|C - C^{\text{prev}}| < \varepsilon$$

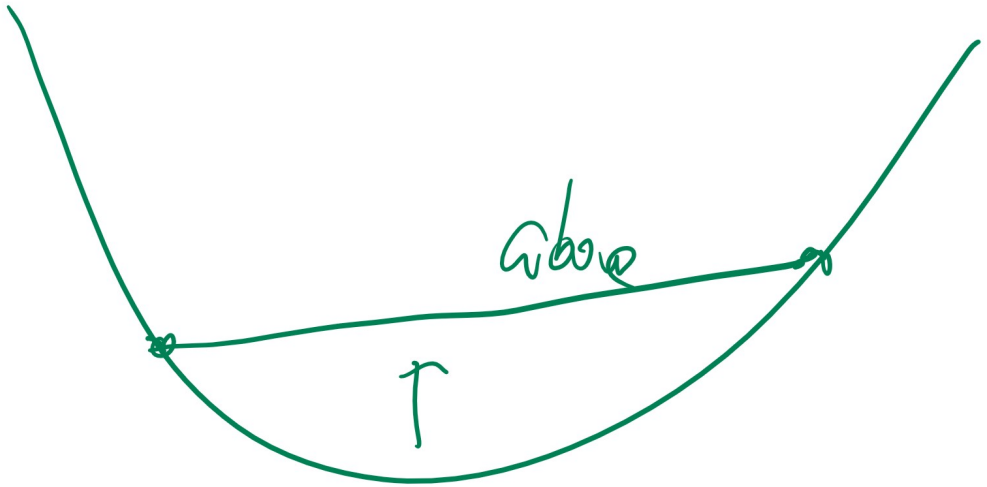
Other Non-linear Activation Function

Discussion

- Activation function: $g\left(\boxed{\cdot}\right) = \tanh\left(\boxed{\cdot}\right) = \dfrac{e^{\boxed{\cdot}} - e^{\boxed{\cdot}}}{e^{\boxed{\cdot}} + e^{\boxed{\cdot}}}$
- Activation function: $g\left(\boxed{\cdot}\right) = \arctan\left(\boxed{\cdot}\right)$
- Activation function (rectified linear unit): $g\left(\boxed{\cdot}\right) = \boxed{\cdot}\mathbb{1}_{\left\{\boxed{\cdot}\geqslant 0\right\}}$
- All these functions lead to objective functions that are convex and differentiable. Gradient descent can be used.

Convexity Diagram

Discussion



Convexity

Discussion

- If a function is convex, gradient descent with any initialization will converge to the global minimum.
- If a function is not convex, gradient descent with different initializations may converge to different local minima.
- A twice differentiable function is convex if and only its second derivative is non-negative.
- In the multivariate case, it means the Hessian matrix is positive semidefinite.

Positive Semidefinite

Discussion

- Hessian matrix is the matrix of second derivatives:

$$H : H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$f(x_1, x_2)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

- A matrix H is positive semidefinite if $x^T H x \geqslant 0 \; \forall \; x \in \mathbb{R}^n$.
- A symmetric matrix is positive semidefinite if and only if all of its eigenvalues are non-negative.

Eigenvalues

Quiz (Participation)

- Eigenvalue?
- A: Never heard of it before.
- B: Heard it once in Avengers Endgame.
- C: Heard it before in other courses.
- D: Learned in other courses before completely forgot.
- E: Still remember how to compute eigenvalues.

Convex Functions

Quiz (Participation)

- What is the Hessian (second derivative) of

$$f(x) = \frac{1}{2}\left(x_1^2 + 4x_1x_2 + x_2^2\right) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- A: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
- B: Do not choose this.
- C: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
- D: Do not choose this.
- E: $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

$x^2 + 2x_1x_2 + 2x_1x_2 + x_2^2$

$(x_1 + 2x_2,\ 2x_1 + x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}(2x_1 + 4x_2)\right) = 1$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}(4x_1 + 2x_2)\right) = 2$

Definiteness

Quiz (Participation)

- Which ones (two) of the following are the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$? Two eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- A: 0
- B: 1
- C: 2
- ✓ D: 3
- E: 4

$\lambda = \{-1, 3\}$

$Hx = \lambda x$ (eigenvalue, eigenvector)

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = ?\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

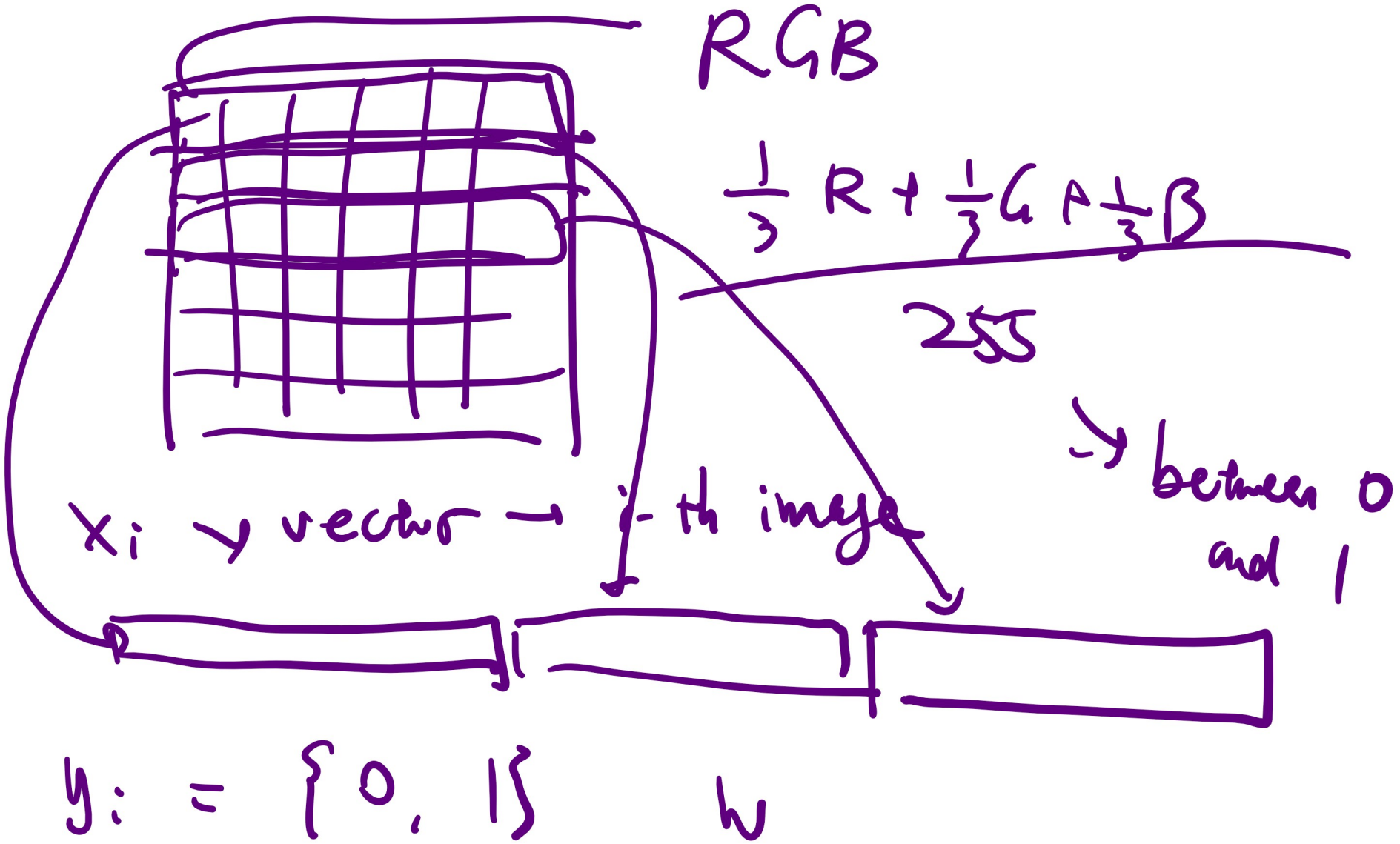
Image as Input

Discussion

- Simplest feature vector for an image is the flattened pixel intensities.
- One way to compute pixel intensity is to use the average of the RGB values divided by 255.
- Pixel intensity of each pixel is between 0 and 1.
- An n_w pixel by n_h pixel image then can be flattened into a $m = n_w n_h$ dimensional input feature vector x.

Flattened Feature Vector Diagram

Discussion



AND Operator Data

Quiz (Particpation)

- Sample data for AND

x_1	x_2	y
0	0	0
0	1	0
1	0	0
1	1	1

Learning AND Operator

Quiz (Particpation)

- Which one of the following is AND?
- A: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-1.5\geqslant 0\}}$
- B: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-0.5\geqslant 0\}}$
- C: $\hat{y} = \mathbb{1}_{\{-1x_1+0.5\geqslant 0\}}$
- D: $\hat{y} = \mathbb{1}_{\{-1x_1-1x_2+0.5\geqslant 0\}}$
- E: None of the above

OR Operator Data

Quiz (Graded)

- Sample data for OR

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	1

Learning OR Operator

Quiz (Graded)

- Which one of the following is OR?
- A: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-1.5\geqslant 0\}}$
- B: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-0.5\geqslant 0\}}$
- C: $\hat{y} = \mathbb{1}_{\{-1x_1+0.5\geqslant 0\}}$
- D: $\hat{y} = \mathbb{1}_{\{-1x_1-1x_2+0.5\geqslant 0\}}$
- E: None of the above

XOR Data

Quiz (Graded)

- Sample data for XOR

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

Learning XOR Operator

Quiz (Graded)

- Which one of the following is XOR?
- A: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-1.5\geqslant 0\}}$
- B: $\hat{y} = \mathbb{1}_{\{1x_1+1x_2-0.5\geqslant 0\}}$
- C: $\hat{y} = \mathbb{1}_{\{-1x_1+0.5\geqslant 0\}}$
- D: $\hat{y} = \mathbb{1}_{\{-1x_1-1x_2+0.5\geqslant 0\}}$
- E: None of the above