CS 540: Introduction to Artificial Intelligence Summer 2020

Final Review

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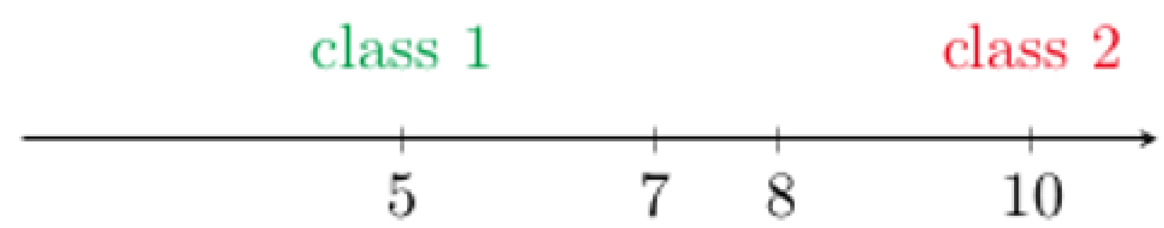
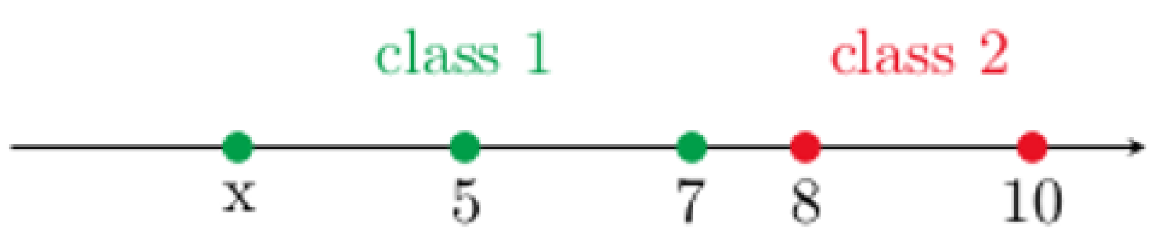
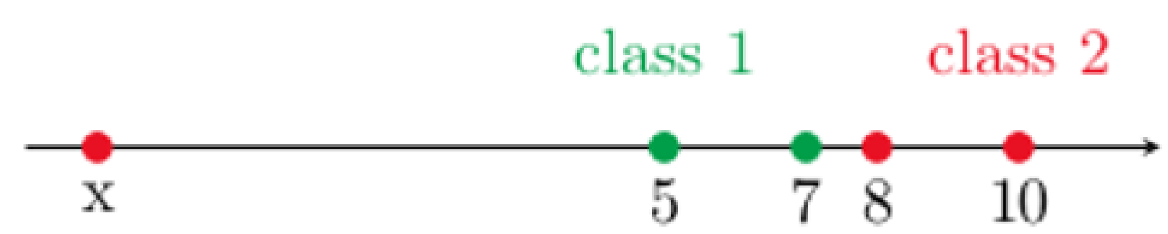
1. **(Summer 2019 Sample Final C Q4)** Suppose K-Means with $K = 2$ is used to cluster the dataset $\{5, 7, 8, 10, x\}$ and initial cluster centers $c_1 = 5, c_2 = 10$. What is x if one of the cluster centers in the next iteration is 5? Here, x can belong to either cluster 1 or 2.



Solution: 7 belongs to class 1 since $d(c_1, 7) < d(c_2, 7)$; 8 belongs to class 2 since $d(c_1, 8) > d(c_2, 8)$. Therefore, class 1 contains $\{5, 7\}$; class 2 contains $\{8, 10\}$.



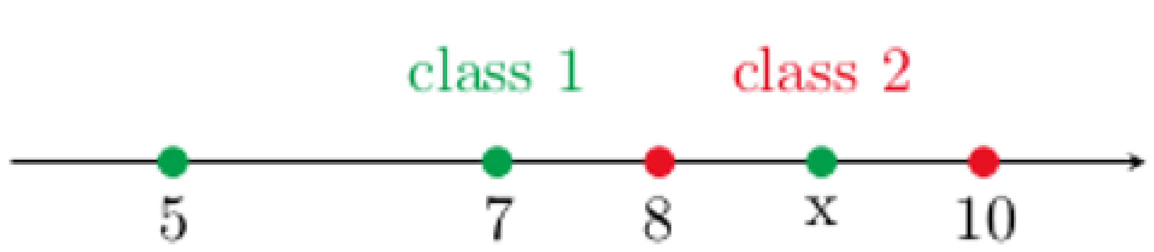


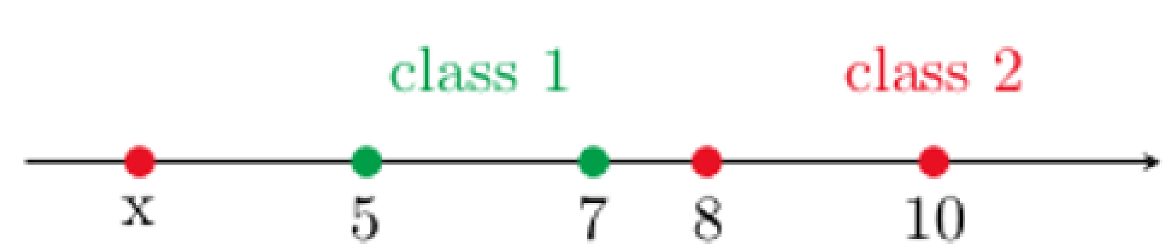
- If x belongs to class 1, updated cluster center $c_2 = \frac{8+10}{2} = 9$ is determined. Then cluster center of class 1 has to be 5, i.e $\frac{x+5+7}{3} = 5$. So $x = 3$, as shown on the left.
- If x belongs to class 2, updated cluster center $c_1 = \frac{5+7}{2} = 6$ is determined. Then cluster center of class 2 has to be 5, i.e $\frac{x+8+10}{3} = 5$. So $x = -3$, as shown on the right. But $d(c_1, -3) < d(c_2, -3)$, which means x cannot belong to class 2 when $x = -3$.

In other words, $x = 3$ is the only answer.

2. **(Summer 2019 Sample Final C Q5)** Continue from the previous question (but ignore the last sentence). What is x if one of the cluster centers in the next iteration is 7? Here, x can belong to either cluster 1 or 2.

Solution:





- If x belongs to class 1, updated cluster center $c_2 = \frac{8+10}{2} = 9$ is determined. Then cluster center of class 1 has to be 7, i.e $\frac{x+5+7}{3} = 7$. So $x = 9$, as shown on the left. But $d(c_1, 9) > d(c_2, 9)$, which means x cannot belong to class 1 when $x = 9$.
- If x belongs to class 2, updated cluster center $c_1 = \frac{5+7}{2} = 6$ is determined. Then cluster center of class 2 has to be 7, i.e $\frac{x+8+10}{3} = 7$. So $x = 3$, as shown on the right. But $d(c_1, 3) < d(c_2, 3)$, which means x cannot belong to class 2 when $x = 3$.

In other words, it is impossible to find a proper x in this case.

3. **(M8Q8) (Spring 2017 Midterm Q4)** You are given the distance table. Consider the next iteration of hierarchical agglomerative clustering (another name for the hierarchical clustering method we covered in the lectures) using complete linkage. What will the new values be in the resulting distance table corresponding to the four new clusters? If you merge two columns (rows), put the new distances in the column (row) with the smaller index. For example, if you merge columns 2 and 4, the new column 2 should contain the new distances and column 4 should be removed, i.e. the columns and rows should be in the order (1), (2 and 4), (3), (5).

$$d = \begin{bmatrix} 0 & 86 & 63 & 78 & 1 \\ 86 & 0 & 22 & 15 & 77 \\ 63 & 22 & 0 & 17 & 43 \\ 78 & 15 & 17 & 0 & 30 \\ 1 & 77 & 43 & 30 & 0 \end{bmatrix}$$

Hint: the resulting matrix should have 4 columns and 4 rows.

Solution:

	A	B	C	D	E
A	0	86	63	78	1
B	86	0	22	15	77
C	63	22	0	17	43
D	78	15	17	0	30
E	1	77	43	30	0

	AE	B	C	D
AE	0	?	?	?
B	?	0	22	15
C	?	22	0	17
D	?	15	17	0

	AE	B	C	D
AE	0	77	43	30
B	77	0	22	15
C	43	22	0	17
D	30	15	17	0

single linkage

$$d(AE, B) = \mathbf{min}\{d(B,A), d(B,E)\} = \min\{86, 77\} = 77$$
$$d(AE, C) = \mathbf{min}\{d(C,A), d(C,E)\} = \min\{63, 43\} = 43$$
$$d(AE, D) = \mathbf{min}\{d(D,A), d(D,E)\} = \min\{78, 30\} = 78$$

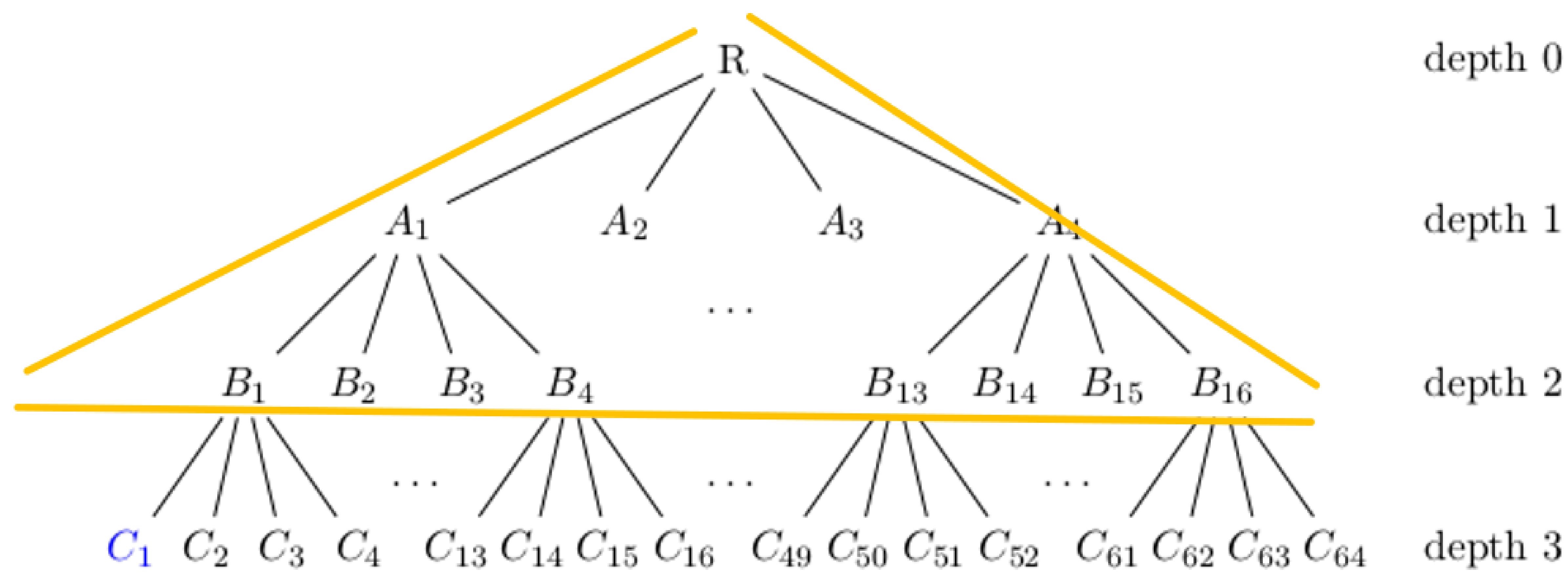
	AE	B	C	D
AE	0	86	63	78
B	86	0	22	15
C	63	22	0	17
D	78	15	17	0

complete linkage

$$d(AE, B) = \mathbf{max}\{d(B,A), d(B,E)\} = \max\{86, 77\} = 86$$
$$d(AE, C) = \mathbf{max}\{d(C,A), d(C,E)\} = \max\{63, 43\} = 63$$
$$d(AE, D) = \mathbf{max}\{d(D,A), d(D,E)\} = \max\{78, 30\} = 78$$

4. **(M9Q7)(Fall 2017 Final Q14)** Consider a search graph which is a tree, and each internal node has 4 children. The only goal node is at depth 3 (root is depth 0). How many total goal-checks will be performed by Iterative Deepening Search in the luckiest case (i.e. the smallest number of goal-checks)? If a node is checked multiple times you should count that multiple times.

Solution:



- Breadth First Search (BFS)
  - $R \;\; A_1 \;\; \ldots \;\; A_4 \;\; B_1 \;\; \ldots \;\; B_{16} \;\; C_1$
  - $4^0 + 4^1 + 4^2 + 1$
- Depth First Search (DFS)
  - $R \;\; A_1 \;\; B_1 \;\; C_1$
  - $1 + 1 + 1 + 1$
- Iterative Deepening Search (IDS): $3 \cdot 4^0 + 2 \cdot 4^1 + 4^2 + 4$
  - DFS at depth 0
    * R
    * 4^0
  - DFS at depth 0 and 1
    * $R \;\; A_1 \;\; A_2 \;\; A_3 \;\; A_4$
    * $4^0 + 4^1$
  - DFS at depth 0, 1 and 2
    * $R \;\; A_1 \;\; B_1 \;\; \ldots \;\; B_4 \;\; A_2 \;\; B_5 \;\; \ldots \;\; B_8 \;\; A_3 \;\; B_9 \;\; \ldots \;\; B_{12} \;\; A_4 \;\; B_{13} \;\; \ldots \;\; B_{16}$
    * $4^0 + 4^1 + 4^2$
  - DFS at depth 0, 1, 2 and 3
    * $R \;\; A_1 \;\; B_1 \;\; C_1$
    * $1 + 1 + 1 + 1$

5. **(M10Q2, M10Q3)(Fall 2018 Midterm Q6, Fall 2017 Midterm Q10)** Let h_1 be an admissible heuristic from a state to the optimal goal, A* search with which ones of the following h will **(never)** be admissible?

(A) $h(n) = h_1(n) \cdot 2$

(B) $h(n) = h_1(n)^2$

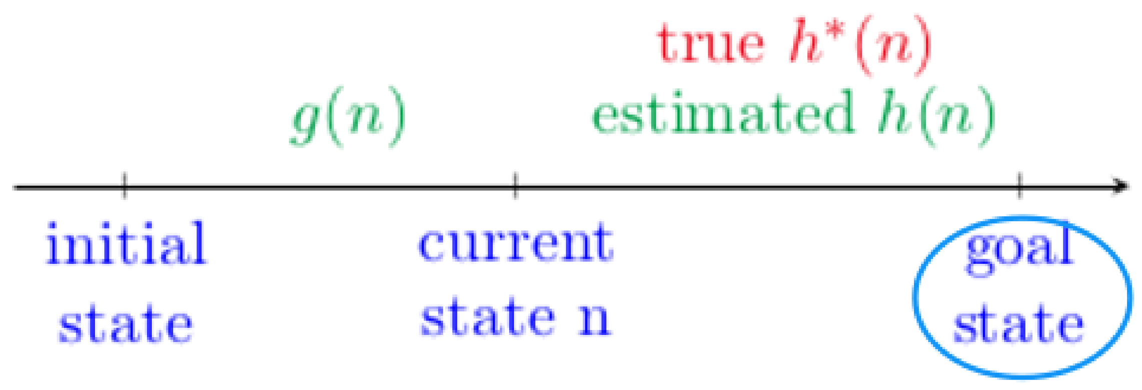
(C) $h(n) = \sqrt{h_1(n)}$

(D) $h(n) = h_1(n) + 1$

(E) $h(n) = h_1(n) - 1$

(F) $h(n) = \frac{h_1(n)}{2}$

Solution: A heuristic is admissible if it never over estimates the true cost, i.e. $0 \leq h(n) \leq h^*(n)$, where $h(n)$ is the admissible heuristic and $h^*(n)$ is the true cost. When h is admissible, $h(n) = 0$ if $n =$ goal state because $0 \leq h(\text{goal}) \leq h^*(\text{goal}) = 0$.



$0 \leq h_1(n) \leq h^*(n)$ since $h_1(n)$ is admissible.

- When $h \leq h_1$
  - If $h < 0 \leq h_1 \leq h^*$, $h(n)$ is not admissible.
  - If $0 \leq h \leq h_1 \leq h^*$, $h(n)$ is admissible.
    * For goal state, we have $0 \leq h(\text{goal}) \leq h_1(\text{goal}) \leq h^*(\text{goal}) = 0$, i.e. $h(\text{goal}) = 0$.
    * If $h_1(n) = 0$, given $0 \leq h \leq h_1 = 0$, we have $h(n) = 0$.

if h1= h*:
0 <= h <= h1 = h*

- When $h > h_1$
  - If $0 \leq h_1 < h^* < h$, $h(n)$ is not admissible.
  - If $0 \leq h_1 < h \leq h^*$, $h(n)$ is not always admissible.
    * For goal state, $h(\text{goal}) > h_1(\text{goal}) = 0 = h^*(\text{goal})$, i.e. $h(n) > h^*(n)$.
    * If $h_1(n) = 0$, then $h \geq h1 = 0$.

In other words,

- When $h < 0$ or $h > h^*$, $h(n)$ is never admissible.
- When $0 \leq h \leq h_1 \leq h^*$, $h(n)$ is always admissible.

(A) $h(n) = h_1(n) \cdot 2$

- If $h_1(n) = 0$, $h(n) = 2h_1(n) = 0 = h_1(n)$.
- If $h_1(n) \neq 0$, $h(n) = 2h_1(n) > h_1(n)$.

i.e. It cannot be decided whether $h(n)$ is never admissible or always admissible.

(B) $h(n) = h_1(n)^2$

- If $h_1(n) > 1$, $h_1 < h_1^2 = h$, then $h(\text{goal}) > h_1(\text{goal}) = 0 = h^*(\text{goal})$.
- If $h_1(n) \leq 1$, $h_1 \geq h_1^2 = h$, then $0 \leq h \leq h_1 \leq h^*$.

i.e. It cannot be decided whether $h(n)$ is never admissible or always admissible.

(C) $h(n) = \sqrt{h_1(n)}$

- If $h_1(n) \geq 1$, $h_1 \geq \sqrt{h_1} = h$, then $0 \leq h \leq h_1 \leq h^*$.
- If $h_1(n) < 1$, $h_1 < \sqrt{h_1} = h$, then $h(\text{goal}) > h_1(\text{goal}) = 0 = h^*(\text{goal})$.

i.e. It cannot be decided whether $h(n)$ is never admissible or always admissible.

(D) $h(n) = h_1(n) + 1$
Since $h = h_1 + 1 > h_1$, i.e. $h > h_1$, $h(n)$ is never admissible given $h(\text{goal}) > h_1(\text{goal})$ $= 0 = h^*(\text{goal})$, i.e. $h(n) > h^*(n)$.

(E) $h(n) = h_1(n) - 1$
Given $h = h_1 - 1 < h_1$, we have $h < h_1$. However, $h(\text{goal}) = h_1(\text{goal}) - 1 = 0 - 1 = -1$ < 0. Therefore, $h(n)$ is never admissible.

(F) $h(n) = \frac{h_1(n)}{2}$
Since $0 \leq \frac{h_1(n)}{2} \leq \frac{h^*(n)}{2} \leq h^*(n)$, i.e. $0 \leq h(n) \leq h^*(n)$, then $h(n)$ is admissible.

In conclusion,

- will be admissible: (F)
- will never be admissible: $(D)(E)$

6. **(Summer 2019 Sample Final B Q26)** Suppose the states are given by five integers from $\{0, 1\}$. The fitness function is the position of the last 1 in the sequence, i.e. $F(x_1, x_2, x_3, x_4, x_5) = \max\{t \in \{0, 1, 2, 3, 4, 5\} : x_t = 1\}$, with $x_0 = 1$. There are in total five states. Using the genetic algorithm, the reproduction probability of first state a is $\frac{1}{5}$. What is the reproduction probability of the last state e? The reproduction probabilities are proportional to the fitness of the states.

$$x_a = (0, 0, 1, 0, 0)$$
$$x_b = (0, 1, 0, 0, 1)$$
$$x_c = (1, 0, 1, 1, 0)$$
$$x_d = (0, 0, 0, 0, 0)$$
$$x_e =?$$

Solution:

	x_0	x_1	x_2	x_3	x_4	x_5	fitness
x_a	1	0	0	1	0	0	$\text{argmax}\{x_0, x_3\} = \max\{0, 3\} = 3$
x_b	1	0	1	0	0	1	$\text{argmax}\{x_0, x_2, x_5\} = \max\{0, 2, 5\} = 5$
x_c	1	1	0	1	1	0	$\text{argmax}\{x_0, x_1, x_3, x_4\} = \max\{0, 1, 3, 4\} = 4$
x_d	1	0	0	0	0	0	$\text{argmax}\{x_0\} = \max\{0\} = 0$

Reproduction probability of first state a is

$$p_a = \frac{F(x_a)}{F(x_a) + F(x_b) + F(x_c) + F(x_d) + F(x_e)} = \frac{3}{3 + 5 + 4 + 0 + F_e} = \frac{1}{5}$$

so $F(x_e) = 3$. Therefore, reproduction probability of last state e is

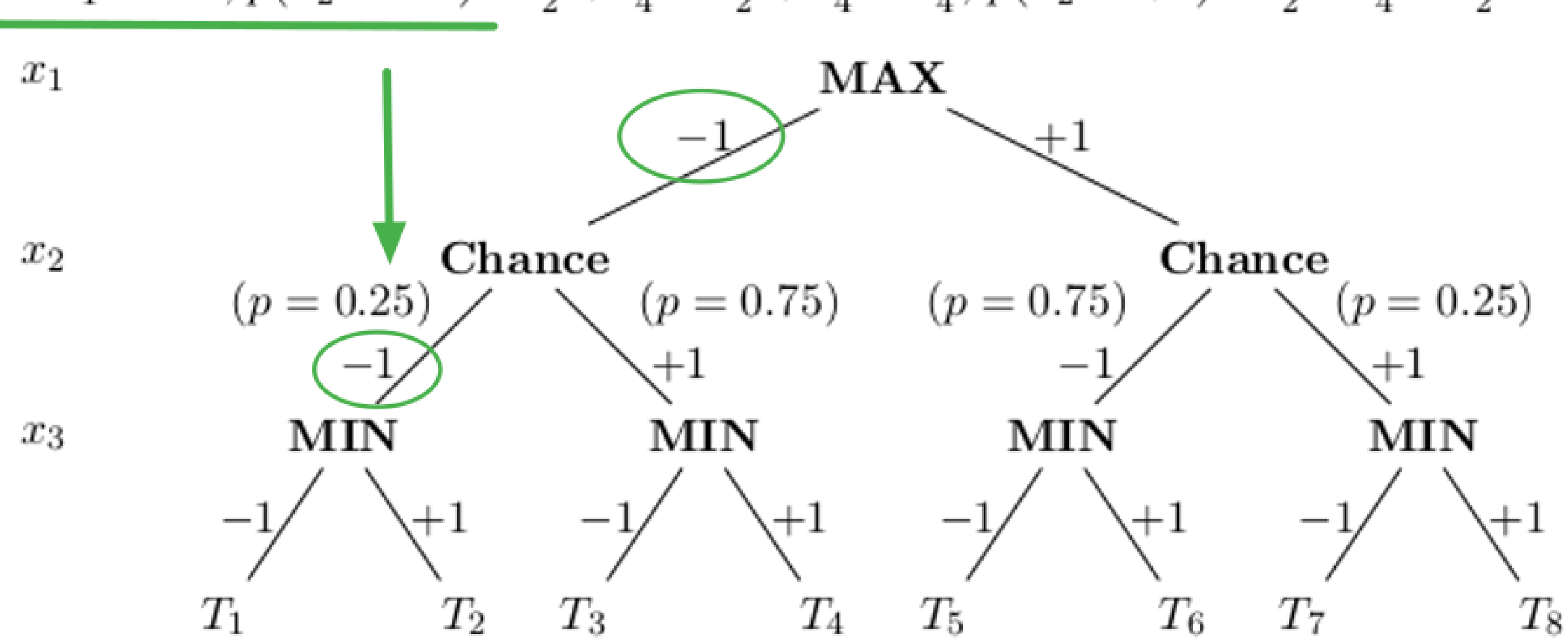
$$p_e = \frac{F(x_e)}{F(x_a) + F(x_b) + F(x_c) + F(x_d) + F(x_e)} = \frac{3}{3 + 5 + 4 + 0 + 3} = \frac{1}{5}$$

7. **(Summer 2019 Final B Q34)** Consider a zero-sum sequential move game with Chance. Player MAX first chooses between actions $x_1 \in \{-1, +1\}$, then Chance chooses $x_2 \in \{-1, +1\}$, $\begin{cases} -1 \text{ with probability } \frac{1}{2} + \frac{x_1}{4} \\ +1 \text{ with probability } \frac{1}{2} - \frac{x_1}{4} \end{cases}$. At the end, player MIN chooses between actions $x_3 \in \{-1, +1\}$. The value of the terminal states corresponding to the actions (x_1, x_2, x_3) is $(x_1 + 2 + x_3) \cdot x_2$. What is the value of the whole game?

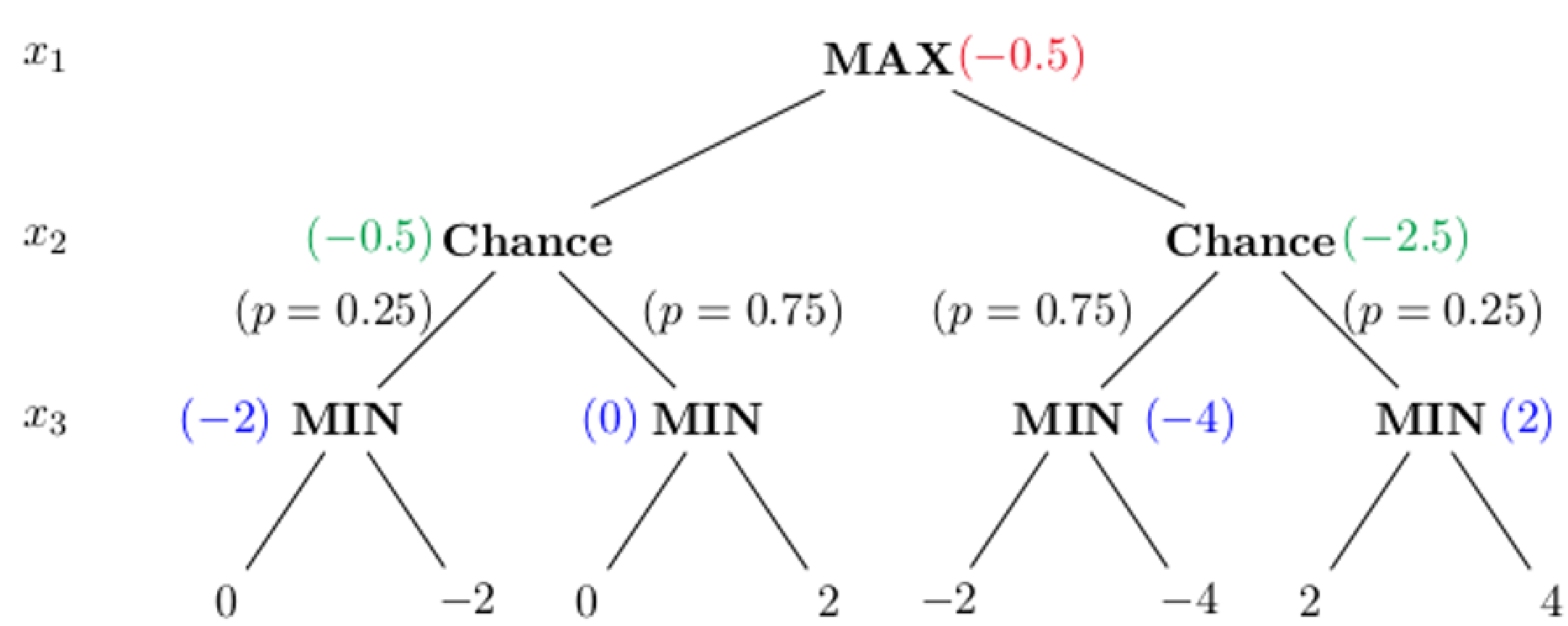
Solution:

When $x_1 = +1$, $p(x_2 = -1) = \frac{1}{2} + \frac{x_1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $p(x_2 = +1) = \frac{1}{2} - \frac{x_1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

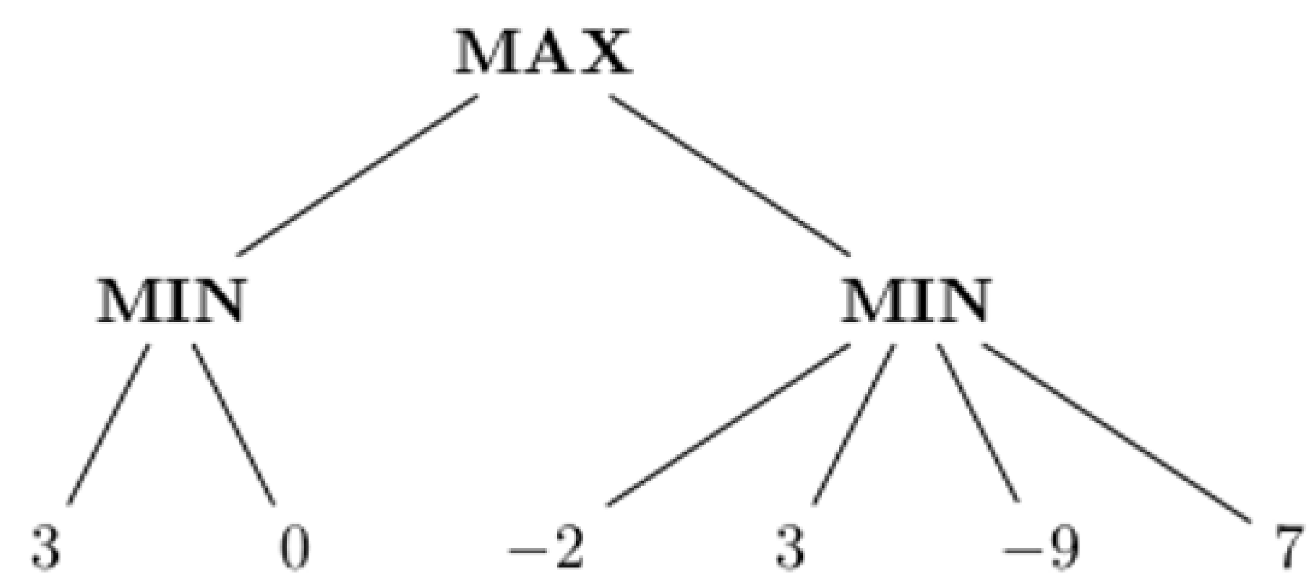
When $x_1 = -1$, $p(x_2 = -1) = \frac{1}{2} + \frac{x_1}{4} = \frac{1}{2} + \frac{-1}{4} = \frac{1}{4}$, $p(x_2 = +1) = \frac{1}{2} - \frac{x_1}{4} = \frac{1}{2} - \frac{-1}{4} = \frac{3}{4}$



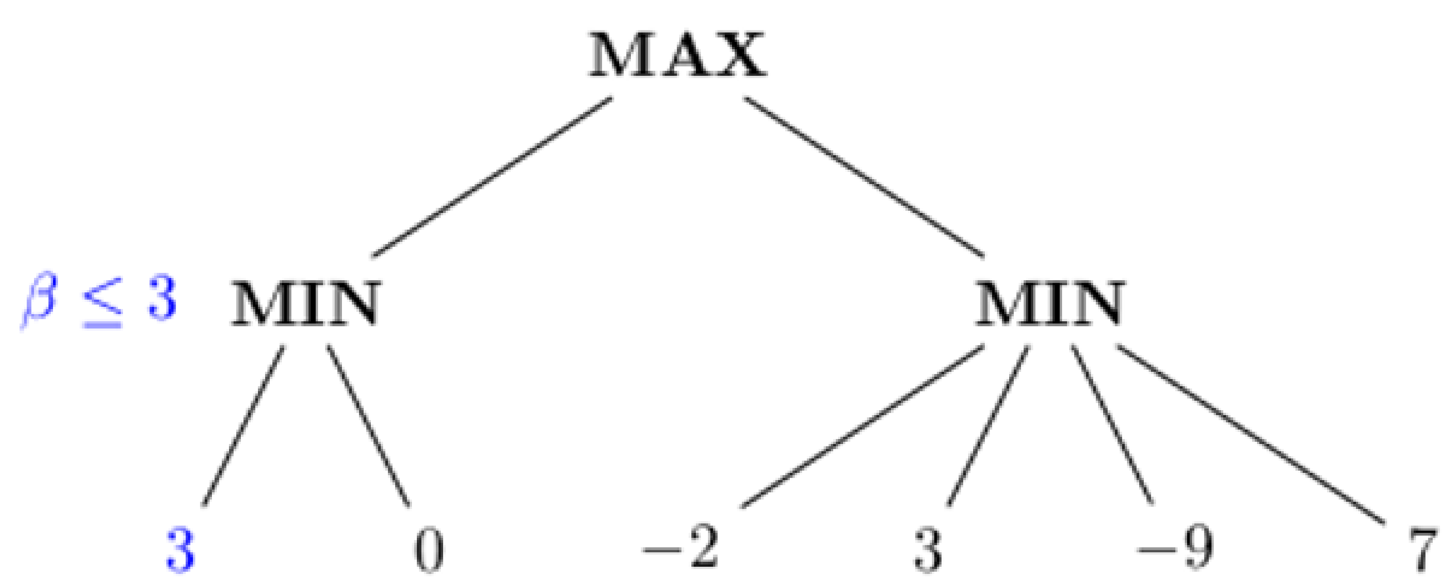
terminal state	action x_3	x_2	x_1	value $(x_1 + 2 + x_3) \cdot x_2$	MIN	Chance	MAX
T_1	-1	-1	-1	0	-2	$-2 \cdot 0.25 + 0 \cdot 0.75 = -0.5$	-0.5
T_2	+1	-1	-1	-2			
T_3	-1	+1	-1	0	0		
T_4	+1	+1	-1	2			
T_5	-1	-1	+1	-2	-4	$-4 \cdot 0.75 + 2 \cdot 0.25 = -2.5$	
T_6	+1	-1	+1	-4			
T_7	-1	+1	+1	2	2		
T_8	+1	+1	+1	4			

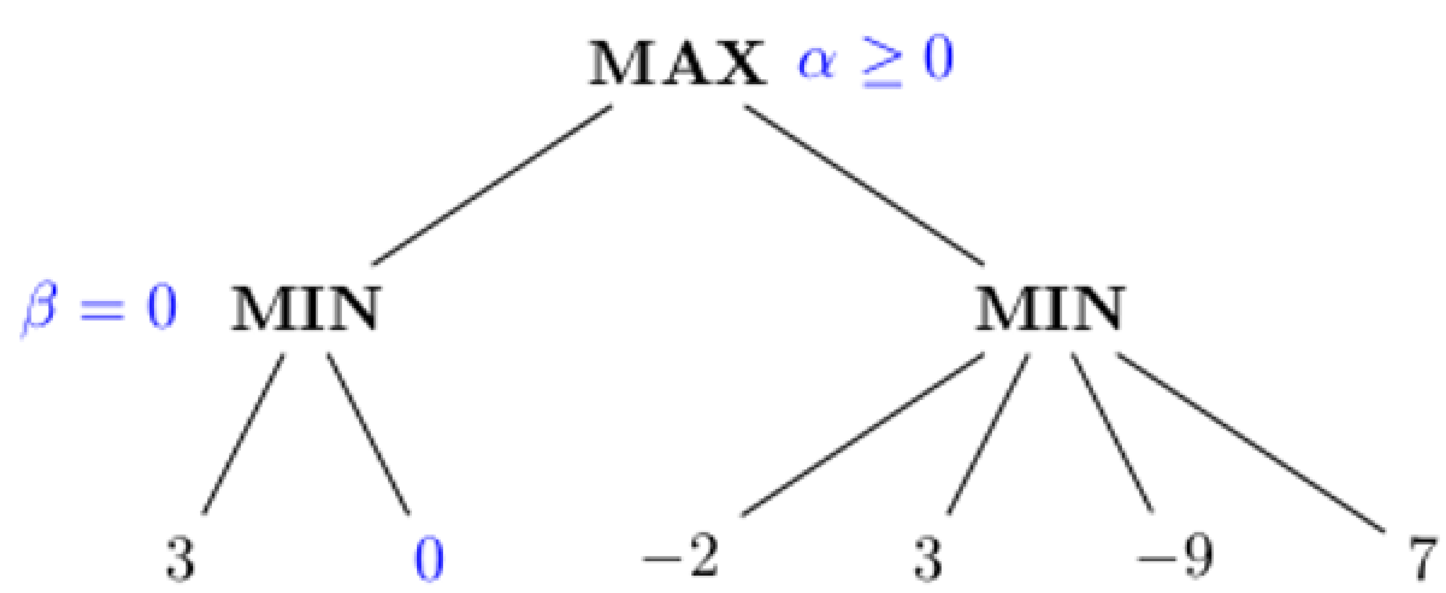


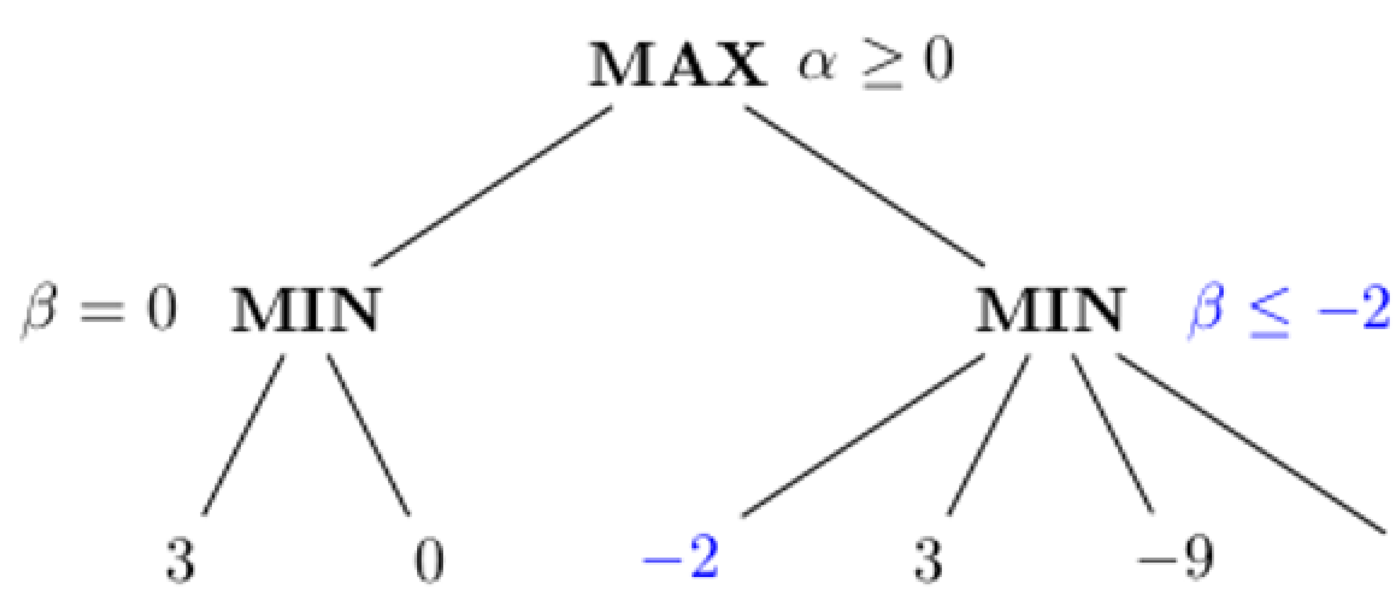
8. **(M12Q2)(Fall 2017 Midterm Q12, Fall 2016 Midterm Q8, Fall 2014 Final Q13, Fall 2012 Final Q17)** Which nodes are pruned by alpha-beta pruning? The max player moves first.



Solution:

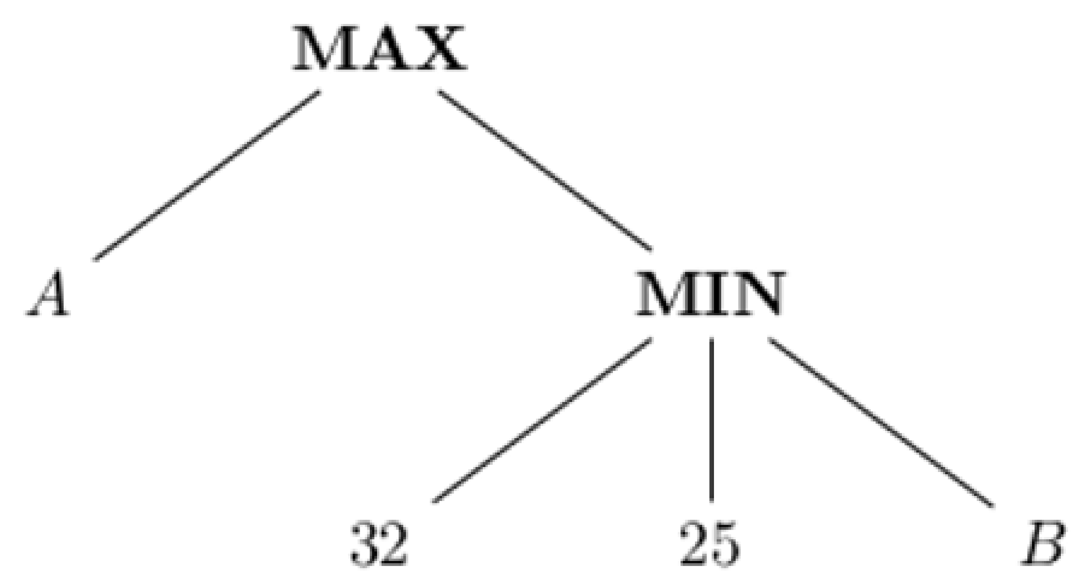




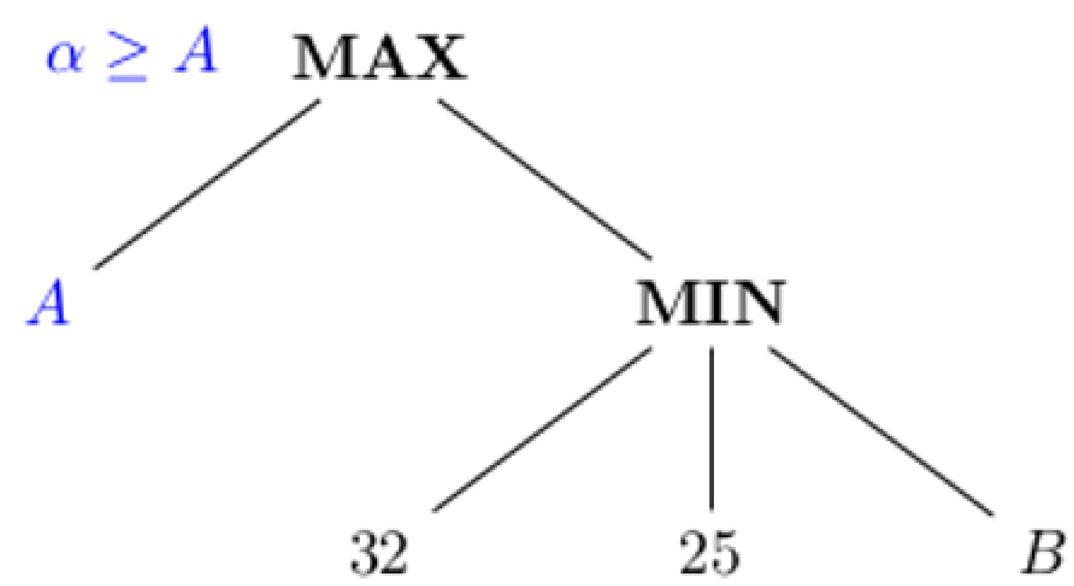


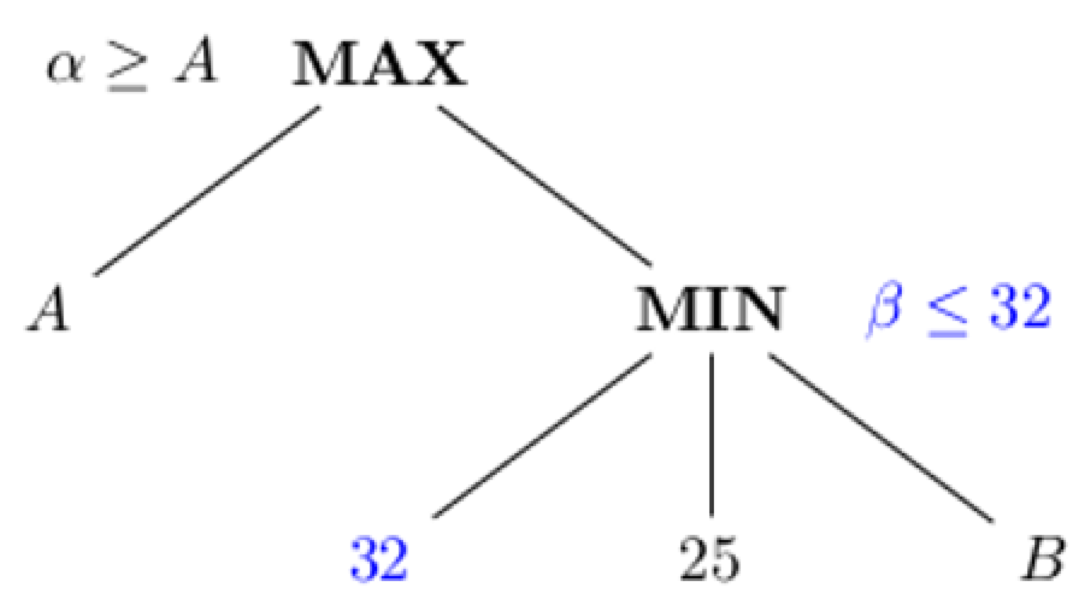
$\alpha \geq \beta$ is satisfied given $\alpha \geq 0$ and $\beta \leq -2$. Therefore, nodes 3, -9, 7 are pruned by alpha-beta pruning.

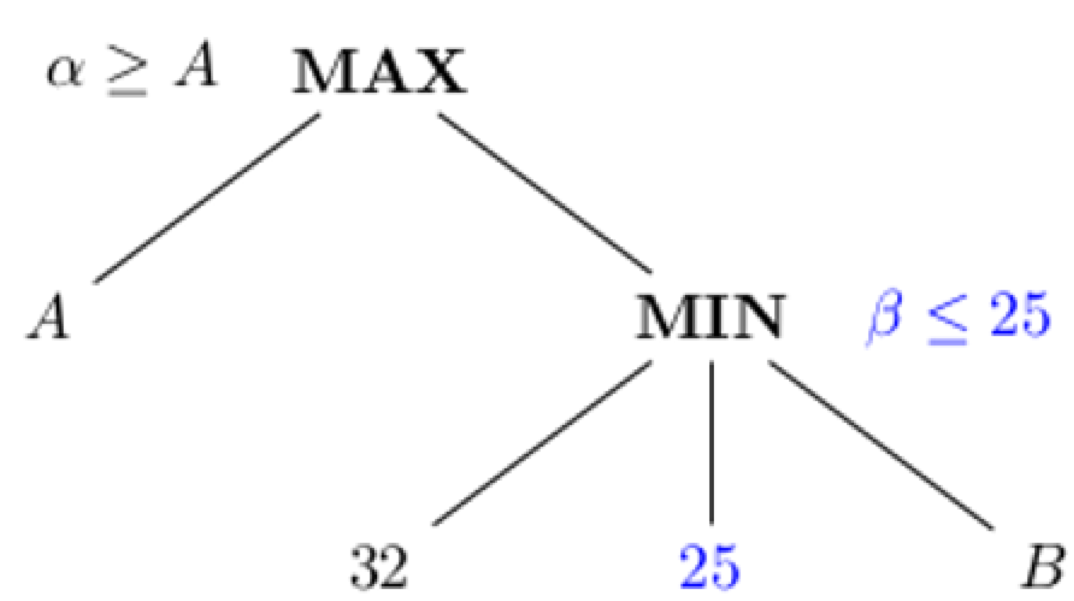
9. **(M12Q1)(Fall 2018 Midterm Q7)** Select the values of A such that B will be alpha-beta pruned. In the case $\alpha = \beta$ (i.e. tie-breaking rule), prune the node.



Solution:







In order to prune B, $\alpha \geq \beta$ should be satisfied, i.e. $A \geq 25$ in this case.

10. **(Summer 2019 Final A Q37)** Given the following simultaneous move game payoff table, what is the complete set of pure strategy Nash equilibria? Both players are MAX players. In each entry of the table, the first number is the reward to the ROW player and the second number is the reward to the COLUMN player.

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

Solution: (Lecture 20 Part 2) If there is a pair (a, b) in a cell in the payoff table, it means the payoff of the ROW player is a and the payoff of the COLUMN player is b.

Fixing COLUMN player:

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

Fixing ROW player:

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

In other words, the complete set of pure strategy Nash equilibrium is $\{(U, L), (D, R)\}$.

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

11. **(Summer 2019 Final A Q38)** Continue from the previous question, how many of the following are mixed strategy Nash equilibria of the game?

$$\left(\text{always } U, \left(L\ \frac{1}{2} \text{ of the time}, R\ \frac{1}{2} \text{ of the time}\right)\right)$$

$$\left(\text{always } D, \left(L\ \frac{1}{2} \text{ of the time}, R\ \frac{1}{2} \text{ of the time}\right)\right)$$

$$\left(\left(U\ \frac{1}{2} \text{ of the time}, D\ \frac{1}{2} \text{ of the time}\right), \text{always } L\right)$$

$$\left(\left(U\ \frac{1}{2} \text{ of the time}, D\ \frac{1}{2} \text{ of the time}\right), \text{always } R\right)$$

$$\left(\left(U\ \frac{1}{2} \text{ of the time}, D\ \frac{1}{2} \text{ of the time}\right), \left(L\ \frac{1}{2} \text{ of the time}, R\ \frac{1}{2} \text{ of the time}\right)\right)$$

Solution:

(A) ROW = always U, COLUMN = $(\frac{1}{2}L, \frac{1}{2}R)$

When ROW = always U:

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

expected payoff(L) = 1
expected payoff(R) = 1
best response$_{\text{COL}}(q)$ = any mix of L and R

COLUMN = $(\frac{1}{2}L, \frac{1}{2}R)$:

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

expected payoff(U) $= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$
expected payoff(D) $= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$
best response$_{\text{ROW}}(p)$ = any mix of U and D

i.e. (always U, $(\frac{1}{2}L, \frac{1}{2}R)$) is a mixed strategy Nash equilibrium.

(B) ROW = always D, COLUMN = $(\frac{1}{2}L, \frac{1}{2}R)$

When ROW = always D:

	L	R
U	$(1, 1)$	$(0, 1)$
D	$(0, 0)$	$(1, 1)$

expected payoff(L) = 0
expected payoff(R) = 1
best response$_{\text{COL}}(q) = R \neq (\frac{1}{2}L, \frac{1}{2}R)$

No need to check whether ROW = always D when COLUMN = $(\frac{1}{2}L, \frac{1}{2}R)$.

i.e. (always D, $(\frac{1}{2}L, \frac{1}{2}R)$) is not a mixed strategy Nash equilibrium.

(C) COLUMN = always L, ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$

When COLUMN = always L:

	L	R
U	$(1,1)$	$(0,1)$
D	$(0,0)$	$(1,1)$

expected payoff(U) = 1
expected payoff(D) = 0
best response$_{\text{ROW}}(p) = U \neq (\frac{1}{2}U,\ \frac{1}{2}D)$

No need to check whether COLUMN = always L when ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$.

i.e. $((\frac{1}{2}U,\ \frac{1}{2}D)$, always $L)$ is not a mixed strategy Nash equilibrium.

(D) COLUMN = always R, ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$

When COLUMN = always R:

	L	R
U	$(1,1)$	$(0,1)$
D	$(0,0)$	$(1,1)$

expected payoff(U) = 0
expected payoff(D) = 1
best response$_{\text{ROW}}(p) = D \neq (\frac{1}{2}U,\ \frac{1}{2}D)$

No need to check whether COLUMN = always R when ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$.

i.e. $((\frac{1}{2}U,\ \frac{1}{2}D)$, always $R)$ is not a mixed strategy Nash equilibrium.

(E) ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$, COLUMN = $(\frac{1}{2}L,\ \frac{1}{2}R)$

When COLUMN = $(\frac{1}{2}L,\ \frac{1}{2}R)$:

	L	R
U	$(1,1)$	$(0,1)$
D	$(0,0)$	$(1,1)$

expected payoff(U) $= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$
expected payoff(D) $= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$
best response$_{\text{ROW}}(p)$ = any mix of U and D

When ROW = $(\frac{1}{2}U,\ \frac{1}{2}D)$:

	L	R
U	$(1,1)$	$(0,1)$
D	$(0,0)$	$(1,1)$

expected payoff(L) $= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$
expected payoff(R) $= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$
best response$_{\text{COL}}(q) = R \neq (\frac{1}{2}L,\ \frac{1}{2}R)$

i.e. $((\frac{1}{2}U,\ \frac{1}{2}D),\ (\frac{1}{2}L,\ \frac{1}{2}R))$ is not a mixed strategy Nash equilibrium.

(Lecture 20 Part 3) As for a mixed strategy Nash equilibrium, suppose COLUMN player chooses L with probability p, then the probability to choose R is $1 - p$.

	p L	$1-p$ R
U	(1, 1)	(0, 1)
D	(0, 0)	(1,1)

For ROW player:
expected payoff(U) $= 1 \cdot p + 0 \cdot (1-p) = p$
expected payoff(D) $= 0 \cdot p + 1 \cdot (1-p) = 1 - p$

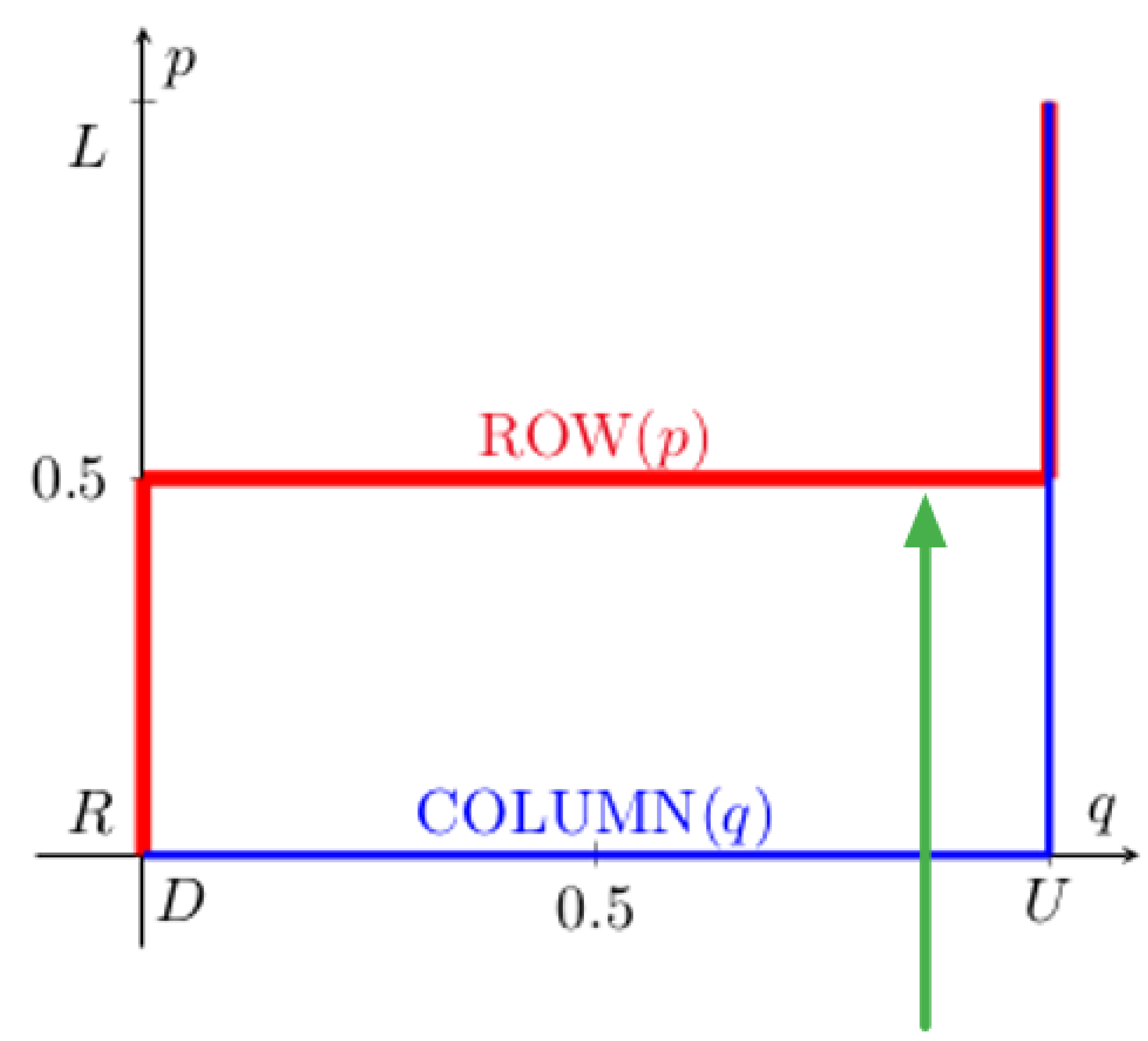
$$\text{best response}_{\text{ROW}}(p) = \begin{cases} U, \text{if } p > 1 - p \Rightarrow p > \frac{1}{2} \\ \text{any mix of } U \text{ and } D, \text{if } p = \frac{1}{2} \\ D, \text{if } p < 1 - p \Rightarrow p < \frac{1}{2} \end{cases} \tag{1}$$

Suppose ROW player chooses U with probability q, then the probability to choose D is $1 - q$.

		L	R
q	U	(1, 1)	(0, 1)
$1-q$	D	(0, 0)	(1, 1)

For COLUMN player:
expected payoff(L) $= 1 \cdot q + 0 \cdot (1-q) = q$
expected payoff(R) $= 1 \cdot q + 1 \cdot (1-q) = 1$

$$\text{best response}_{\text{COLUMN}}(q) \begin{cases} \sout{L, \text{if } q > 1} \\ \text{any mix of } L \text{ and } R, \text{if } q = 1 \\ R, \text{if } q < 1 \end{cases} \tag{2}$$



12. **(Summer 2019 Final A Q39)** Continue from the previous question, how many of the following are mixed strategy Nash equilibria of the game?

$$\left(\text{always } U, \left(L\ \frac{1}{4} \text{ of the time}, R\ \frac{3}{4} \text{ of the time}\right)\right)$$

$$\left(\text{always } U, \left(L\ \frac{3}{4} \text{ of the time}, R\ \frac{1}{4} \text{ of the time}\right)\right)$$

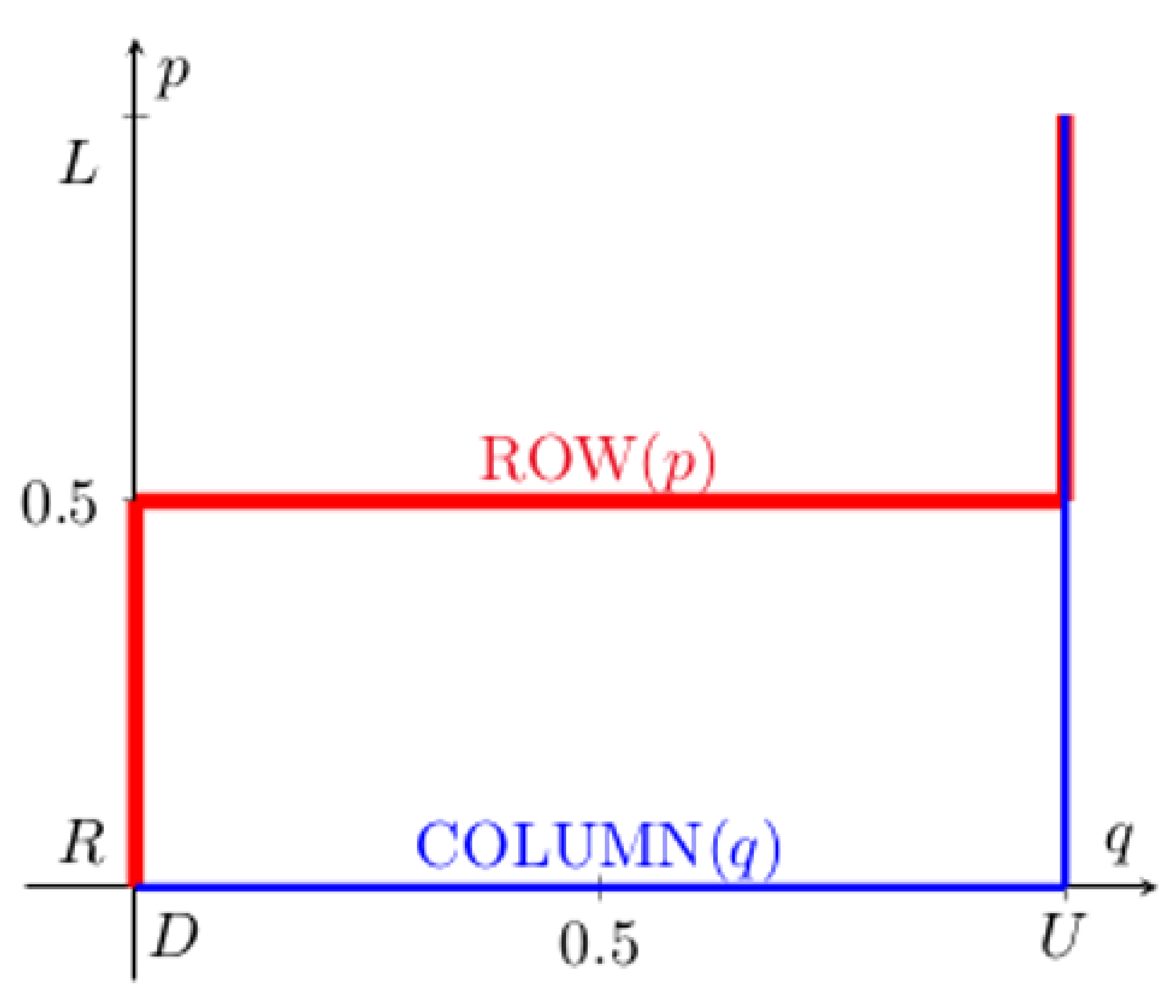
$$\left(\text{always } D, \left(L\ \frac{1}{4} \text{ of the time}, R\ \frac{3}{4} \text{ of the time}\right)\right)$$

$$\left(\text{always } D, \left(L\ \frac{3}{4} \text{ of the time}, R\ \frac{1}{4} \text{ of the time}\right)\right)$$

Solution:

$$\text{COLUMN} = \begin{cases} L, \text{prob} = p \\ R, \text{prob} = 1-p \end{cases} \qquad \text{best response}_{\text{ROW}}(p) = \begin{cases} U, \text{if } p > \frac{1}{2} \\ \text{any mix of } U \text{ and } D, \text{if } p = \frac{1}{2} \\ D, \text{if } p < \frac{1}{2} \end{cases}$$

$$\text{ROW} = \begin{cases} U, \text{prob} = q \\ D, \text{prob} = 1-q \end{cases} \qquad \text{best response}_{\text{COLUMN}}(q) = \begin{cases} \cancel{L, \text{if } q > 1} \\ \text{any mix of } L \text{ and } R, \text{if } q = 1 \\ R, \text{if } q < 1 \end{cases}$$



In other words, $\left(\text{always } U, \left(L\ \frac{3}{4} \text{ of the time}, R\ \frac{1}{4} \text{ of the time}\right)\right)$ is the only mixed strategy Nash equilibrium.

13. **(Summer 2019 Final A Q40)** Continue from the previous question, how many of the following are mixed strategy Nash equilibria of the game?

$$\left(\text{always } U, \left(L \ \frac{1}{8} \text{ of the time}, R \ \frac{7}{8} \text{ of the time}\right)\right)$$

$$\left(\text{always } U, \left(L \ \frac{3}{8} \text{ of the time}, R \ \frac{5}{8} \text{ of the time}\right)\right)$$

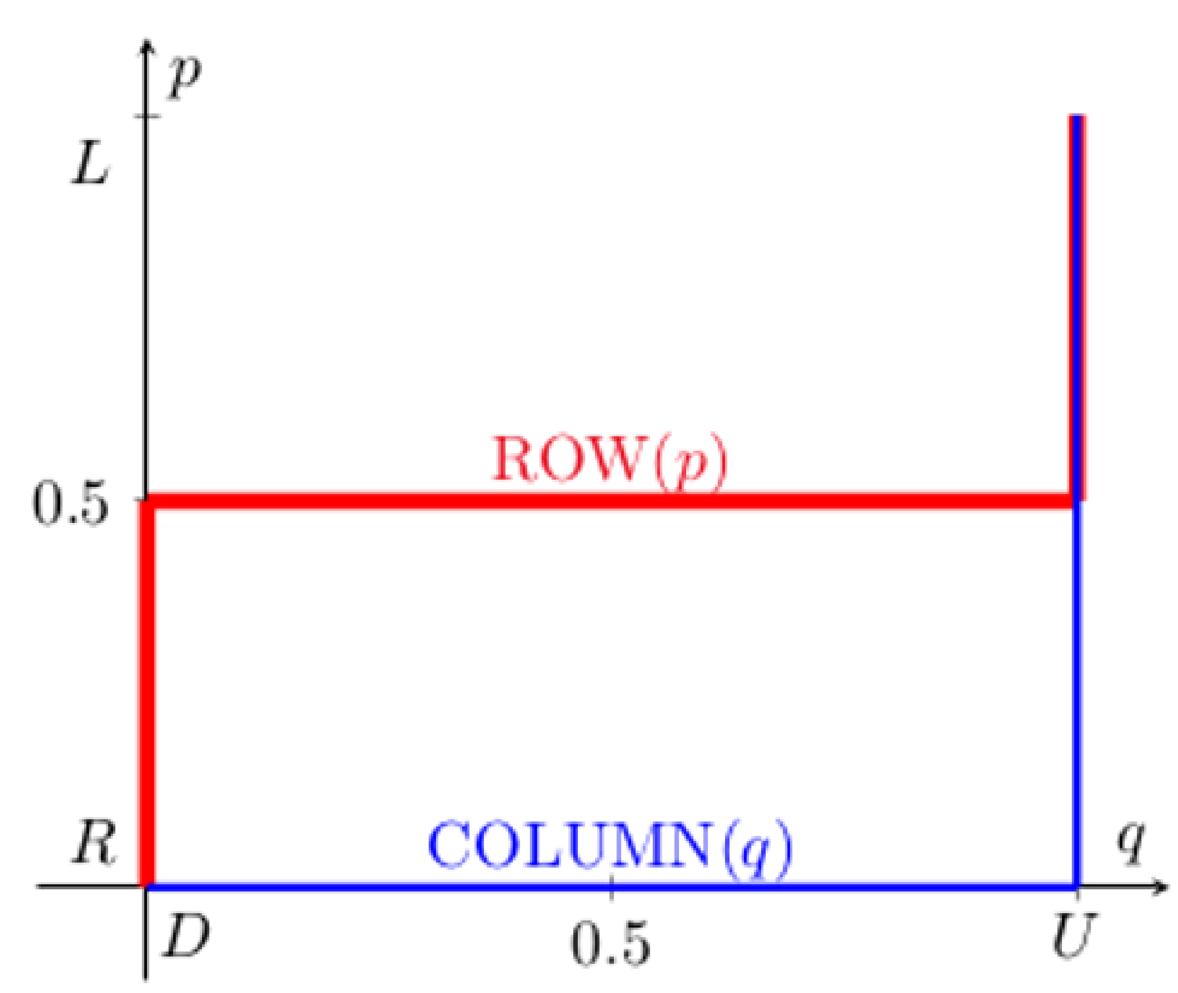
$$\left(\text{always } U, \left(L \ \frac{5}{8} \text{ of the time}, R \ \frac{3}{8} \text{ of the time}\right)\right)$$

$$\left(\text{always } U, \left(L \ \frac{7}{8} \text{ of the time}, R \ \frac{1}{8} \text{ of the time}\right)\right)$$

Solution:

$$\text{COLUMN} = \begin{cases} L, \text{prob} = p \\ R, \text{prob} = 1-p \end{cases} \qquad \text{best response}_{\text{ROW}}(p) = \begin{cases} U, \text{if } p > \frac{1}{2} \\ \text{any mix of } U \text{ and } D, \text{if } p = \frac{1}{2} \\ D, \text{if } p < \frac{1}{2} \end{cases}$$

$$\text{ROW} = \begin{cases} U, \text{prob} = q \\ D, \text{prob} = 1-q \end{cases} \qquad \text{best response}_{\text{COLUMN}}(q) = \begin{cases} \cancel{L, \text{if } q > 1} \\ \text{any mix of } L \text{ and } R, \text{if } q = 1 \\ R, \text{if } q < 1 \end{cases}$$



In other words, $\left(\text{always } U, \left(L \ \frac{5}{8} \text{ of the time}, R \ \frac{3}{8} \text{ of the time}\right)\right)$ and $\left(\text{always } U, \left(L \ \frac{7}{8} \text{ of the time}, R \ \frac{1}{8} \text{ of the time}\right)\right)$ are mixed strategy Nash equilibria.

14. **(Summer 2019 Final B Q37)** Given the following game payoff table, what is the complete set of pure strategy Nash equilibria? Both players are MAX players. In each entry of the table, the first number is the reward to the ROW player and the second number is the reward to the COLUMN player.

	L	R
U	$(3, 1)$	$(0, 0)$
D	$(0, 1)$	$(1, 1)$

Solution:

Fixing COLUMN player:

	L	R
U	$(3, 1)$	$(0, 0)$
D	$(0, 1)$	$(1, 1)$

Fixing ROW player:

	L	R
U	$(3, 1)$	$(0, 0)$
D	$(0, 1)$	$(1, 1)$

In other words, the complete set of pure strategy Nash equilibrium is $\{(U, L), (D, R)\}$.

	L	R
U	$(3, 1)$	$(0, 0)$
D	$(0, 1)$	$(1, 1)$

As for a mixed strategy Nash equilibrium, suppose COLUMN player chooses L with probability p, then the probability to choose R is $1-p$.

	p	$1-p$
	L	R
U	(3, 1)	(0, 0)
D	(0, 1)	(1,1)

For ROW player:
expected payoff(U) $= 3 \cdot p + 0 \cdot (1-p) = 3p$
expected payoff(D) $= 0 \cdot p + 1 \cdot (1-p) = 1-p$

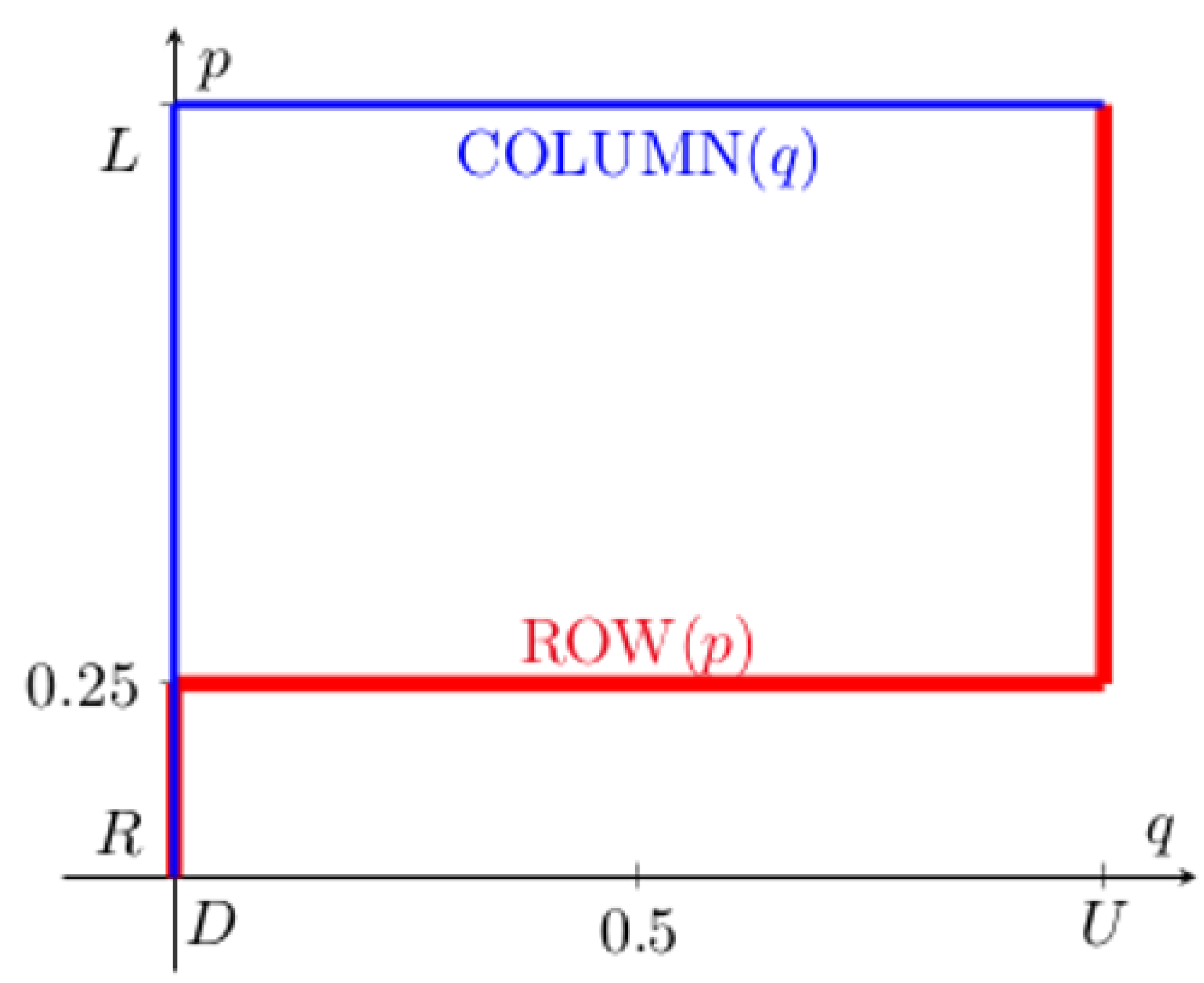
$$\text{best response}_{\text{ROW}}(p) = \begin{cases} U, \text{if } 3p > 1-p \Rightarrow p > \frac{1}{4} \\ \text{any mix of } U \text{ and } D, \text{if } p = \frac{1}{4} \\ D, \text{if } 3p < 1-p \Rightarrow p < \frac{1}{4} \end{cases} \tag{3}$$

Suppose ROW player chooses U with probability q, then the probability to choose D is $1-q$.

		L	R
q	U	(3, 1)	(0, 0)
$1-q$	D	(0, 1)	(1, 1)

For COLUMN player:
expected payoff(L) $= 1 \cdot q + 1 \cdot (1-q) = 1$
expected payoff(R) $= 0 \cdot q + 1 \cdot (1-q) = 1-q$

$$\text{best response}_{\text{COLUMN}}(q) \begin{cases} L, \text{if } q > 0 \\ \text{any mix of } L \text{ and } R, \text{if } q = 0 \\ \text{~~R, if q < 0~~} \end{cases} \tag{4}$$

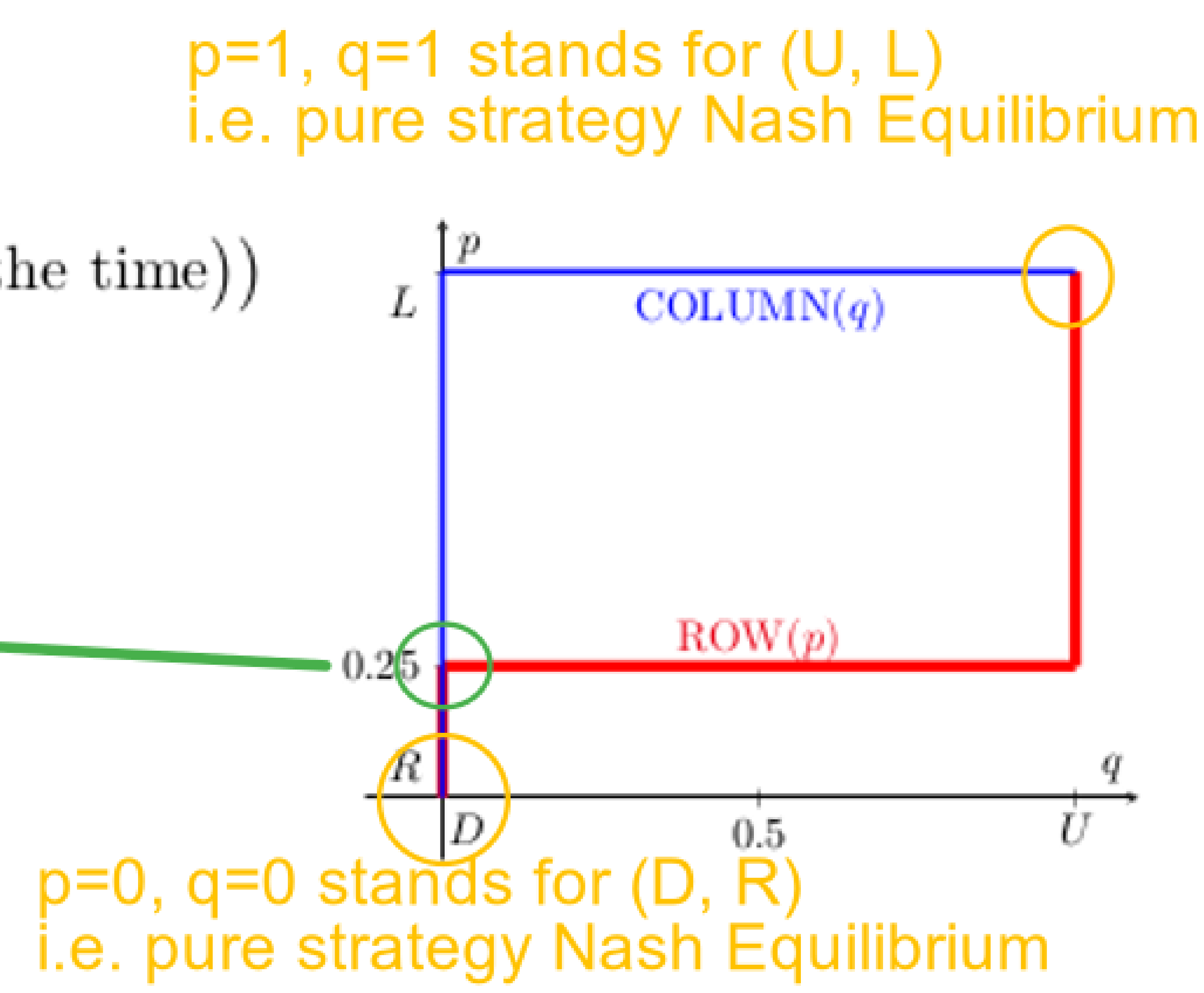


15. **(Summer 2019 Final B Q38-Q40)** Continue from the previous question, how many of the following are mixed strategy Nash equilibria of the game?

$$\left(\text{always } U, \left(L\ \tfrac{1}{2} \text{ of the time}, R\ \tfrac{1}{2} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{1}{2} \text{ of the time}, R\ \tfrac{1}{2} \text{ of the time}\right)\right)$$
$$\left(\left(U\ \tfrac{1}{2} \text{ of the time}, D\ \tfrac{1}{2} \text{ of the time}\right), \text{always } L\right)$$
$$\left(\left(U\ \tfrac{1}{2} \text{ of the time}, D\ \tfrac{1}{2} \text{ of the time}\right), \text{always } R\right)$$
$$\left(\left(U\ \tfrac{1}{2} \text{ of the time}, D\ \tfrac{1}{2} \text{ of the time}\right), \left(L\ \tfrac{1}{2} \text{ of the time}, R\ \tfrac{1}{2} \text{ of the time}\right)\right)$$

$$\left(\text{always } U, \left(L\ \tfrac{1}{4} \text{ of the time}, R\ \tfrac{3}{4} \text{ of the time}\right)\right)$$
$$\left(\text{always } U, \left(L\ \tfrac{3}{4} \text{ of the time}, R\ \tfrac{1}{4} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{1}{4} \text{ of the time}, R\ \tfrac{3}{4} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{3}{4} \text{ of the time}, R\ \tfrac{1}{4} \text{ of the time}\right)\right)$$

$$\left(\text{always } D, \left(L\ \tfrac{1}{8} \text{ of the time}, R\ \tfrac{7}{8} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{3}{8} \text{ of the time}, R\ \tfrac{5}{8} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{5}{8} \text{ of the time}, R\ \tfrac{3}{8} \text{ of the time}\right)\right)$$
$$\left(\text{always } D, \left(L\ \tfrac{7}{8} \text{ of the time}, R\ \tfrac{1}{8} \text{ of the time}\right)\right)$$



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