SCHOOL ALLOCATION PROBLEM WITH OBSERVABLE CHARACTERISTICS

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Abstract. I study a school choice problem where students have observable characteristics that are correlated with their preferences. For example, one such characteristic may be the location of a student's home, which is correlated with preferences if students tend to prefer nearby schools. I consider mechanisms that are envy-free, efficient, and treat students with the same observable characteristics equally. I show that the welfare-maximizing mechanism in this class is a modified probabilistic serial mechanism with capacities. These capacities specify the maximum number of students with given characteristics that can be admitted into each school.

1. Introduction

The literature on school choice studies the problem of how to allocate students into schools. A typical problem includes students with different preferences, and possibly schools with different preferences or priorities for students.

One approach to the school choice problem was introduced in Bogomolnaia and Moulin (2001). They consider a mechanism design approach. Students report their preferences and the mechanism allocates the students into the schools. A good mechanism satisfy certain properties such as the following,

(1) Efficiency: the resulting allocation is not Pareto dominated by any other feasible allocation.
(2) Envy-freeness: the resulting allocation for any individual student is not dominated by the allocation for another student.
(3) Symmetry: the resulting allocation is the same for students with the same preferences.

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Bogomolnaia and Moulin (2001) introduce a mechanism called the simultaneous eating mechanism that is characterized by property (1). They also show that a special case of their simultaneous eating mechanism, called the probabilistic serial mechanism, satisfies all three properties $(1),(2)$ and (3). Later, Liu and Pycia (2016) shows that for problems with full support - meaning that for every possible permutation of the preference ordering over the schools, a strictly positive proportion of students have that preference - a mechanism satisfies $(1),(2)$ and (3) if and only if it is probabilistic serial.

The symmetry assumption made by Bogomolnaia and Moulin (2001) has an important interpretation as fairness criterion: no two students are treated differently. It also has a practical dimension: if the only information about students comes from their reports, only the reports can be taken into account by the mechanism. However, the assumption is not appropriate for many contexts. In many situations, categories of students can be observed by the schooling authority. Examples include,

- The further the location of a student's home to a school, the higher the transportation cost and as a result, ceteris paribus, the less the student will prefer to go that school. The school board may have additional data on the average amount of saving on transportation costs for each student, and as a result have good estimate of the cardinal welfare for every feasible allocation.
- Students with higher standardized test grades may be more likely to enjoy a school with larger libraries, more labs and better teachers. The students may not be able to evaluate the benefit from better libraries, but the school board is able to provide a more accurate estimate of the difference in benefit to students with different test scores.
- The more siblings a student has in a school, the higher the utility the student will obtain when attending that school. In this case, two students with the same ordinal preference ordering may have dramatically different cardinal utilities from attending the schools. It is impossible to obtain from the students the cardinal utility gain from having a sibling in the same school because there are no consistent units of

measurement; however, the school board may be able to evaluate the gain relatively consistently.

- Students with disabilities may benefit more from schools with more specialized facilities, equipment and teachers with training for students with special needs. Similarly to the previous examples, the school board has better information on the utility gain for these students.

In the above situations, there are important reasons to treat different students differently. For instance, the school authority may want to reduce the transportation costs and give the priority to the students who live in the school's neighborhood. Similarly, the priority to schools that are accessible can be given to students with disabilities.

In this paper, I consider a version of the school choice problem where students that belong to different well-defined groups can be treated differently. Formally, as in Bogomolnaia and Moulin (2001), I take the mechanism design approach, and I consider stochastic allocations and compare them through first-order stochastic dominance. I will make the assumption that although the students have cardinal utilities, they can only report ordinal rankings, and as a result, the mechanism is ordinal. For tractability, I also assume continuum of students of mass 1.

I keep the efficiency assumption and replace envy-freeness and symmetry with a pair of weaker assumptions that students with the same characteristic do not envy each other's allocations and they must be treated equally and call these assumptions within-group envy-freeness and within-group symmetry. Consequently, I consider the class of mechanisms that satisfy the following properties:

(1) Efficiency: the resulting allocation is not Pareto dominated by any other feasible allocation
(2) Within-group envy-freeness: the resulting allocation for any individual student is not dominated by the allocation for another student *with the same characteristics* .
(3) Within-group symmetry: the resulting allocation is the same for students *with the same characteristics* with the same preferences.

The main result is similar to the one in Liu and Pycia (2016) that the only mechanisms that satisfy the above three properties are from a class of modified versions of the probabilistic serial mechanisms from Bogomolnaia and Moulin (2001). These mechanisms are probabilistic serial with school-specific subcapacities for each group of students with the same characteristic. Each subcapacity specifies the maximum amount of students with a particular characteristic that are allowed to be assigned to the school. In addition, given data on the expected cardinal utilities from assigning students with a particular characteristic to a specific school, I show that the cardinally efficient allocations can also be obtained from the modified probabilistic serial mechanisms with subcapacities that can be chosen according the solution to a convex programming problem.

Here, I briefly describe the algorithm as if the students are consuming portions of schools continuously: for students in a specific group, schools start with sizes equal to their subcapacities and every student starts eating her favorite school at the same unit rate until some schools are completely eaten. Then, students start to eat their favorite among the remaining schools that are not eaten. The process ends at time 1, and the total amount of school a student has eaten will represent the probability that she is allocated to school.

This paper is most closely related to the literature on ordinal mechanisms for large markets developed by Kojima and Manea (2010), and Che and Kojima (2010) and extended by Liu and Pycia (2016). The proofs also use techniques from these papers. The literature attempts to find efficient, strategy-proof and symmetric mechanisms for large markets; however, the focus of this paper deviates from the literature in that it studies problems in which the students have observable characteristics that can provide additional information for the designers. As a result, this paper partially addresses the loss of welfare due to the restriction to ordinal mechanisms mentioned in Abdulkadiroğlu, Che, and Yasuda (2011) and Pycia (2014) by proposing the use of estimated cardinal utilities (from students' observable characteristics) in addition to the elicited ordinal preference. The results are not directly applicable to finite markets, but the algorithm in this paper can be applied to finite markets and potentially improve welfare in high school choice assignments in Boston or New York described in Pathak and Sethuraman (2011), Abdulkadiroğlu, Pathak, and Roth (2009) and Pathak and Sönmez (2008).

Section 2 sets up the model with ordinal preferences. Section 3 defines the class of modified probabilistic serial mechanisms with subcapacities and explains why they are the only mechanisms that are efficient, within-group envy-free and symmetric. Section 4 sets up the model for the planner with cardinal preferences and explains the convex (linear) programming problem to solve for the optimal subcapacities.

2. Ordinal $Mo\delta$

In this section, I describe the model.

There is a finite set of schools $L = \{1, 2, ..., \bar{L}\}$ and a finite set of characteristics $K = \{1, 2, ..., \bar{K}\}$. Each student has one of these characteristics, thus the students are partitioned into groups by their characteristics. School l has mass of c_l spots and there are mass μ_k of students with characteristic k. The total mass of the students is 1 and I assume that there is enough room in the schools to assign each student,

$$\sum_{l \in L} c_l \geqslant \sum_{k \in K} \mu_k = 1.$$

A student's preference is given by a strict ordering $p \in \mathcal{P}(L)$, where $\mathcal{P}(L)$ is the set of all permutations (strict orderings) of L. A *type* is a pair (k, p), and $\mu(k, p)$ denotes the mass of students in group k who have preference ordering p. Let the individual students be indexed by $i \in \mathcal{I} = [0, 1]$, equipped with the Lebesgue measure λ.

Definition 1. A *(full support) profile* of students is a λ-measurable function, *type* : $\mathcal{I} \to K \times \mathcal{P}(L)$, that satisfies the following conditions:

(1) Type and group consistency,

$$\sum_{p \in \mathcal{P}(L)} \mu(k, p) = \mu_k,$$

where,

$$\mu(k,p) = \int_{i \in \mathcal{I}, type(i)=(k,p)} d\lambda.$$

(2) Full support,

$$\mu(k,p) > 0 \; \forall \; (k,p) \in K \times \mathcal{P}(L).$$

In this definition, a profile maps the students' names to their types, (1) states that the total mass of students with type (k,p) is $\mu(k,p)$ and the total mass of students in group k is μ_k, and (2) states that the measure of students with every possible type is strictly positive. More importantly, each profile induces a full support distribution of types, μ, where $\mu(k,p)$ is the mass of students of type (k,p).

Definition 2. An *allocation* is a measurable function $q : \mathcal{I} \to \Delta L$, where $q\,(l;i)$ is the probability that a student $i \in \mathcal{I}$ is assigned to school l. An allocation q is *feasible* if,

$$\int_{i \in \mathcal{I}} q\,(l;i)\, d\lambda \leqslant c_l.$$

An allocation maps each student in $\mathcal{I}$ to a stochastic allocation. One interpretation is that students receive their spot randomly according the distribution $q\,(i)$. Each stochastic allocation can be implemented by a randomized mechanism. For fixed μ_k, let $\mathcal{M}$ be the set of all profiles and $\mathcal{Q}$ be the set of all feasible allocations.

Definition 3. A *mechanism* is a function $Q : \mathcal{M} \to \mathcal{Q}$, that maps every profile to a feasible allocation.

Next, I define ordinal efficiency using first order stochastic dominance.

Definition 4. An allocation q is *dominated* by q' for a student with type (k,p) if

$$\sum_{l' >_p l} q\,(l';i) \leqslant \sum_{l' >_p l} q'\,(l';i) \;\; \forall \; l \in L,$$

with strict inequality for at least one l.

One allocation dominates another if it first order stochastically dominates the other allocation. The three desired properties of an ordinal mechanism in this model are the following. I will use the notation $k : \mathcal{I} \to K$ to represent the function that maps a student to her group characteristics.

Definition 5. Efficiency: $q(i)$ is not dominated by any $q'(i)$ for any $i \in \mathcal{I}$.

Definition 6. Within-group envy-free: $q(i)$ is not dominated by any $q(j)$ for any $i, j \in \mathcal{I}$ such that $k(i) = k(j)$.

Definition 7. Within-group symmetry: for each type $(k, p), q(i) = q(j)$ for any $i, j \in \mathcal{I}$ such that $type(i) = type(j) = (k, p)$.

Within-group symmetry states that students with the same characteristics and preferences should be assigned the same allocation. Given this assumption, I can use the notation $q(l; k, p)$ to denote the probability of a student with type (k, p) getting allocated the school l. Envy-freeness states that any a student (k, p) will not prefer the allocation of another student (k, p') for $p' \neq p$. The assumption is a weaker version of strategy-proofness. Efficiency implies that no other allocation is preferred by all students.

In the case in which every student with the same type gets the same allocation, the notation $q(l; k, p)$ will be used in place of $q(l; i)$ to denote the probability that a student with type (k, p) is assigned to school l. Consequently, the total amount of students with type (k, p) who are assigned to school l satisfies,

$$\mu(k, p) \cdot q(l; k, p) = \int_{i \in \mathcal{I} : type(i) = (k, p)} q(l; i) \, d\lambda.$$

3. Modified Probabilistic Serial

In this section, I describe the probabilistic serial mechanism from Bogomolnaia and Moulin (2001) and the modification to include subcapacities that represent the maximum number of students with a certain characteristic that can be allocated to each school. I also state and explain the main result that a mechanism is efficient, within-group envy-free and within-group symmetric if and only if it is modified probabilistic serial.

I start by briefly summarizing the mechanism in Bogomolnaia and Moulin (2001). I describe their algorithm as if the students are consuming portions of schools continuously at a fixed rate. Schools start with sizes equal to their capacities and every student starts eating her favorite school at rate 1 until some schools are completely eaten. Then, students start to eat their favorite among the remaining schools that are not eaten. The process ends at time 1, and the total amount of school a student has eaten will represent the probability that she is allocated to school.

Then I define the collection of subcapacities $\left\{c_l^k\right\}_{k \in K, l \in L}$. Subcapacity c_l^k represents the maximum number of students in group k that can be assigned to school l by the algorithm. They must satisfy the feasibility condition,

$$\sum_{k \in K} c_l^k \leqslant c_l \; \forall \; l \in L.$$

I modify the mechanism in Bogomolnaia and Moulin (2001) in two ways.

(1) Schools start with sizes equal to their capacities but are divided into subcapacities for each group specified by $\left\{c_l^k\right\}_{l \in L, k \in K}$, so the initial sizes of the schools are not equal to their actual capacities.

(2) Students in one group cannot eat the portion of the schools allocated to other groups.

In this algorithm, all students with the same type get the same allocation. Therefore, within-group symmetry is ensured. The formal descriptions of the algorithms are as follows. Before that, I define the following function.

$M(l, A)$ is a function that maps a school, l, and a set of available schools, $A \subseteq L$, to the set of preference orderings in which the favorite school among the set A is l,

$$M(l, A) = \left\{p \in \mathcal{P}(L) : l \succ l' \; \forall \; l' \in A \backslash \{l\}\right\}. \tag{1}$$

Algorithm 1. *Given subcapacities $\left\{c_l^k\right\}_{l\in L,k\in K}$, the probabilistic serial mechanism (PS) assigns to each profile the allocation resulting from the following process, assuming μ is the distribution of types induced by the given profile.*

Initialize: $L_k^0 = L, y_k^0 = 0$ for each $k \in K$ and $q^0(l;k,p) = 0$ for each $l \in L, k \in K, p \in \mathcal{P}(L)$,

Iteration: Assume $L_k^{s-1}, y_k^{s-1}, q^{s-1}$ are defined for each k. For each k,

(1) *For each l, find $y_k^s(l)$, the earliest time at which students in group k finish consuming school l,*

$$y_k^s(l) = \arg\min_y \left\{ \sum_{p\in M\left(l,L_k^{s-1}\right)} \mu(k,p)\left(y - y_k^{s-1}\right) + \sum_{p\in\mathcal{P}(L)} \mu(k,p)\, q^{s-1}(l;k,p) = c_l^k \right\}.$$

(2) *Find y_k^s, the earliest time at which students in group k finish consuming any school,*

$$y_k^s = \min_l y_k^s(l).$$

(3) *Find F_k^s, the set of schools that are completely consumed by students in group k,*

$$F_k^s = \arg\min_l y_k^s(l).$$

(4) *Find the remaining set of available schools for students in group k,*

$$L_k^s = L_k^{s-1} \backslash F_{k.}^s$$

(5) *Find $q^s(l;k,p)$, the temporary allocation of school l for students with type (k,p), which represents the amount of school l the students have eaten so far until the end of step s,*

$$q^s(l;k,p) = q^{s-1}(l;k,p) + \mathbb{1}_{p\in M\left(l,L_k^{s-1}\right)}\left(y^s - y^{s-1}\right).$$

For a full support profile, Bogomolnaia and Moulin (2001) showed that probabilistic serial generates an allocation is efficient, envy-free, symmetric, and Liu and Pycia (2016) showed that an allocation is efficient, envy-free, and symmetric if and only if it is generated by probabilistic serial. The result can be extended to the problem with multiple groups with a similar proof to Theorem 1 in Liu and Pycia (2016).

Proposition 1. *An allocation, q, is efficient, within-group symmetric, and envy-free for a full support profile that induces type distribution μ if and only if it is generated by Algorithm 1 modified probabilistic serial with subcapacities,*

$$c_l^k(\mu) = \sum_{p \in \mathcal{P}(L)} \mu(k,p)\, q(l;k,p).$$

The above formula for c_l^k says nothing about how to choose the subcapacities in practice since they are just one of many capacities that are compatible with the allocation q.

4. Cardinal $Mo\delta$

In this section, I introduce the model with cardinal utility functions that is ordinally consistent with the previous model. I explain that the cardinally efficient allocation can be obtained from the modified probabilistic serial mechanism with subcapacities which can be found as the solution to a convex (linear) optimization problem.

The cardinal utility functions should be compatible with the preference relations from ordinal model. Let $u(i) : L \to \mathbb{R}_+$ be the utility function of a student i, where $u(l;i)$ represents the utility from attending school l. A utility function induces a preference relation p if for every student i with the preference p,

$$l \succ_p l' \text{ whenever } u(l;i) > u(l';i) \;\; \forall\, l \in L.$$

Here, I do not need to assume the students with the same type (k,p) have the same utility function, but since the within-group symmetry assumption requires the allocation to be the same for students with the same type, and the welfare function uses the average utilities, I can restrict attention to using the average utilities over the same type,

$$u(l;k,p) = \frac{1}{\mu(k,p)} \int_{type(i)=(k,p),p(u(i))=p} u(l;i)\, d\lambda,$$

where $p(u)$ is the preference relation induced by the utility ranking u.

Definition 8. A utility distribution that is *consistent* with a preference profile μ is the set of utility functions $u(i)$, such that,

$$\int_{type(i)=(k,p),p(u(i))=p} d\lambda = \mu(k,p) \ \forall\ k \in K, p \in \mathcal{P}(L).$$

Definition 9. The allocation $q^\star$ is *cardinally efficient* if it maximizes the average expected welfare:

$$q^\star \in \arg\max_{q \in \mathcal{Q}} \int_{i \in \mathcal{I}} \sum_{l \in L} u(l;i) \cdot q(l;i)\, d\lambda.$$

If the mechanism is restricted to be within-group symmetric, then $q^\star$ maximizes the welfare function

$$W(q) = \sum_{(k,p) \in K \times \mathcal{P}(L)} \sum_{l \in L} u(l;k,p) \cdot q(l;k,p) \cdot \mu(k,p).$$

Since probabilistic serial is ordinally efficient, any cardinally efficient allocation must be obtained by probabilistic serial for some subcapacities. Therefore, I write the allocation and welfare as a function of the subcapacities in the probabilistic serial mechanism that generates them, let $c = \left\{c_l^k\right\}_{l \in L, k \in K}, c^k = \left\{c_l^k\right\}_{l \in L}$,

$$W_k\left(c^k\right) = \sum_{l \in L} u(l;k,p)\, q^{\text{PS}(c)}(l;k,p)\, \mu(k,p),$$

and,

$$W(c) = \sum_{k \in K} W_k\left(c^k\right),$$

where $q^{\text{PS}(c)}$ is the allocation generated by probabilistic serial with subcapacities c.

Proposition 2. *The function $W(c)$ is concave in c.*

In order to find welfare-maximizing mechanism, I need to find the maximum of function $W(c)$ subject to the following linear constraints.

$$\max_{c_l^k \in [0, c_l]} W(c)$$

$$\text{such that } \sum_{k \in K} c_l^k = c_l$$

$$\text{and } \sum_{l \in L} c_l^k = \mu_k$$

The first constraint is the school capacity constraint requiring the subcapacities for a school for all groups add up to the total physical capacity of the school. The second constraint is the profile constraint requiring the subcapacities for a group from all schools add up to the total mass of the students in that group. Both constraints hold at optimum because of the assumption that there is enough room in the schools to allocate all the students.

The optimization can be done computationally. Proposition 2 says that function $W(c)$ is concave in c, hence, standard gradient methods can be used to find the maximum.

5. Proofs

5.1. **Proof of Proposition 1.** I divide the proof into three parts, stated as the following three lemmas.

Lemma 1. *Modified probabilistic serial is (ordinally) efficient.*

Lemma 2. *Modified probabilistic serial is envy-free.*

Lemma 3. *Given a full-support profile, μ, if an allocation q is efficient, within-group envy-free and symmetric, then it is generated by modified probabilistic serial with constraints $c_l^k = \sum_{p \in \mathcal{P}(L)} q(l; k, p)\, \mu(k, p)$.*

Lemma 1 and Lemma 2 are modified from Theorem 1 and Proposition 1 of Bogomolnaia and Moulin (2001), respectively, and Lemma 3 is modified from Theorem 1 of Liu and

Pycia (2016). The modified probabilistic serial mechanism is within-group symmetric by construction, so these three lemmas, together with the full support assumption, implies Proposition 1.

Proof of Lemma 1: Suppose, for a contradiction that q is obtained by modified probabilistic serial and it is not efficient, and q is dominated by q'.

Let (k, p_1) be the student such that $q(k, p_1) \neq q'(k, p_1)$, there are l_0, l_1 with $l_0 \succ_{p_1} l_1$, such that,

$$q(l_1; k, p_1) > q'(l_1; k, p_1),$$

$$q(l_0; k, p_1) < q'(l_0; k, p_1).$$

Then $l_0 \succ_{p_1} l_1$ and $q(l_1; k, p_1) > 0$.

Similarly, there is $l_1 \succ_{p_2} l_2$ and $q(l_2; k, p_2) > 0$.

Since L is finite, there exists some cycle,

$$l_0 \succ_{p_1} l_1 ... l_R \succ_{p_R} l_{0,}$$

such that for every $r \in \{0, 1, ..., R\}$,

$$l_{r-1} \succ_{p_r} l_r \text{ and } q(l_r; k, p_r) > 0.$$

Take an arbitrary r, let the q^s represent the partial probabilistic serial allocation at time s, and define the following,

$$s_r = \inf_s \{s : q^s(l_r; k_r, p_r) > 0\}.$$

Note that $p_{r-1} \notin L_{k_r}^{s_{r-1}}$, where the notation L_k^s is introduced in step (4) of Algorithm 1 as the remaining set of available schools for students in group k after step s. This implies $s_{r-1} < s_{r.}$

This is true for for every r, implying $s_0 < s_1 < ... < s_{R-1} < s_0$, which leads to a contradiction. □

Proof of Lemma 2: Fix a student (k, p) with $l_1 \succ_p l_2 ... \succ_p l_{\bar{L}}$, and let s_1 be the earliest time at which l_1 is completely eaten, meaning,

$$l_1 \in L_k^{s_1 - 1} \backslash L_k^{s_1}.$$

For $s \leqslant s_1 - 1, (k, p) \in M\left(l_1, L_k^s\right)$, where $M(l, A)$ is introduced before Algorithm 1 by Equation 1 as the set of preference orderings in which the favorite school among the set A is l, and

$$q^{s_1}\left(l_1; k, p\right) = y_k^{s_1} \geqslant q^{s_1}\left(l_1; k, p'\right) \ \forall \ p' \neq p.$$

Since l_1 is completely eaten at s_1, the previous relationship holds for final allocation q as well,

$$q\left(l_1; k, p\right) \geqslant q\left(l_1; k, p'\right) \ \forall \ p' \neq p.$$

Now, let s_2 be the earliest time $\{l_1, l_2\}$ is completely eaten, $s_2 \geqslant s_1$ and for the same reason,

$$q\left(l_1; k, p\right) + q\left(l_2; k, p\right) = y_k^{s_2} \geqslant q\left(l_1; k, p'\right) + q\left(l_2; k, p'\right) \ \forall \ p' \neq p.$$

Repeat the argument to see that $q(k, p)$ dominates $q(k, p')$ for every $p' \neq p$. □

Proof of Lemma 3: Fix any allocation, q', that is efficient, within-group envy-free and symmetric, and the allocation, q^1, obtained by modified probabilistic serial. Let q^t denote the partial allocation at time $t \in [0, 1]$ from the probabilistic serial.

I show that for any student with type $(k, p) \in K \times \mathcal{P}(L)$, any school $l \in L$, and at any time $t \in [0, 1], q'$ dominates q^t. Then, by the efficiency property of q^1 from Lemma 1, $q' = q^1$, which concludes the proof.

Assume for a contradiction, let τ be the earliest time when q' does not dominate q^t. Using the notations in Definition 4, for some school l and and student with type (k, p),

$$\tau = \inf\left\{t : \sum_{l' \succ_p l} q'\left(l'; k, p\right) < \sum_{l' \succ_p l} q^t\left(l'; k, p\right)\right\}.$$

By continuity of the function q^t in t, q' dominates q^t for each $t \in [0, \tau]$. In particular, at time τ, the students with type (k, p) must be eating some school l, and,

$$\sum_{l' >_p l} q'(l'; k, p) \geqslant \sum_{l' >_p l} q^\tau(l'; k, p) = \tau.$$

Since τ the earliest time the above inequality stops holding,

$$\sum_{l' >_{k,p} l} q'(l'; k, p) = \sum_{l' >_{k,p} l} q^\tau(l'; k, p) = \tau.$$

Now, using the full support condition, l must be the favorite object of some agent (k, p'), and,

$$q'(l; k, p') \geqslant q^\tau(l; k, p') = \tau.$$

The envy-free assumption implies (k, p) does not prefer the allocation of (k, p'),

$$q'(l; k, p') \leqslant \tau.$$

Therefore,

$$q'(l; k, p') = \tau.$$

Since school l is not completely eaten at time τ,

$$q^1(l; k, p') > \tau.$$

Therefore, the equalities imply that (k, p') gets less l in q' than q^1. The efficiency of q^1 implies that there is another student $(k, \hat{p})$ who gets,

$$q'(l; k, \hat{p}) > q^1(l; k, \hat{p}).$$

And there is some school $\hat{l}$ that is not l that student $(k, \hat{p})$ prefers just more than school l,

$$\sum_{l' >_{\hat{p}} \hat{l}} q'(l'; k, \hat{p}) \geqslant \sum_{l' >_{\hat{p}} \hat{l}} q^{\tau}(l'; k, \hat{p}),$$

implying,

$$\sum_{l' >_{\hat{p}} \hat{l}} q'(l'; k, \hat{p}) \geqslant \tau - q^{\tau}(l; k, \hat{p}),$$

and,

$$\sum_{l' >_{\hat{p}} \hat{l}} q'(l'; k, \hat{p}) \geqslant \tau - q^{1}(l; k, \hat{p}),$$

and finally,

$$\sum_{l' >_{\hat{p}} l} q'(l'; k, \hat{p}) > \tau.$$

Comparing students (k, p') and $(k, \hat{p})$, envy-freeness of q' leads to a contradiction. □

5.2. **Proof of Proposition 2.** I prove several smaller lemmas about properties of envy-free allocations in order to prove concavity.

Lemma 4. *Any convex combination of envy-free allocations is envy-free.*

Lemma 5. *Any inefficient envy-free allocation has an envy-free Pareto improvement.*

Lemma 6. *The set of envy-free allocations are closed.*

Proof of Lemma 4: Consider arbitrary pair of students (k, p) and (k, p') under two different allocations q_1 and q_2.

Let $p = l_1 \succ l_2 \succ l_3 ... \succ l_{\bar{L}}$ be the preference ranking of the first student, and define the following,

$$t_1^i = \min_t \left\{ \sum_{s=1}^{t} q_1 (l_s; k, p) < \sum_{s=1}^{t} q_1 (l_s; k, p') \right\}$$

$$t_1^a = \max_t \left\{ \sum_{s=1}^{t} q_1 (l_s; k, p) \leqslant \sum_{s=1}^{t} q_1 (l_s; k, p') \right\}$$

$$t_2^i = \min_t \left\{ \sum_{s=1}^{t} q_2 (l_s; k, p) < \sum_{s=1}^{t} q_2 (l_s; k, p') \right\}$$

$$t_2^a = \max_t \left\{ \sum_{s=1}^{t} q_2 (l_s; k, p) \leqslant \sum_{s=1}^{t} q_2 (l_s; k, p') \right\}$$

By envy-freeness, $t_1^i \neq t_1^a$ and $t_2^i \neq t_{2,}^a$

Consider a convex combination $q_0 = (\alpha) q_1 + (1 - \alpha) q_2$ for $\alpha \in [0, 1]$,

For $t \leqslant \min \{t_1^i, t_2^i\}$,

$$\begin{aligned} \sum_{s=1}^{t} q_0 (l_s; k, p) &= \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p) + (1 - \alpha) q_2 (l_s; k, p) \\ &< \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p') + (1 - \alpha) q_2 (l_s; k, p') \\ &= \sum_{s=1}^{t} q_0 (l_s; k, p') \end{aligned}$$

And for $t \geqslant \max \{t_1^a, t_2^a\}$,

$$\begin{aligned} \sum_{s=1}^{t} q_0 (l_s; k, p) &= \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p) + (1 - \alpha) q_2 (l_s; k, p) \\ &> \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p') + (1 - \alpha) q_2 (l_s; k, p') \\ &= \sum_{s=1}^{t} q_0 (l_s; k, p') \end{aligned}$$

Therefore, under q_0, no student strictly prefers the allocation of another student, q_0 is envy-free. □

Proof of Lemma 5: Consider an allocation q and another allocation q' that (Pareto) dominates q.

For each student (k,p) and pair of schools l_1 and l_2, define the flow from school l_1 to l_2 by $\Delta\left(l_1,l_2;k,p\right)$ satisfying:

$$\sum_{l_2\in L}\Delta\left(l_1,l_2;k,p\right)=\max\left\{0,q\left(l_1;k,p\right)-q'\left(l_1;k,p\right)\right\},$$
$$\sum_{l_1\in L}\Delta\left(l_1,l_2;k,p\right)=\max\left\{0,q\left(l_2;k,p\right)-q'\left(l_2;k,p\right)\right\},$$
$$\Delta\left(l_1,l_2;k,p\right)\geqslant 0.$$

Then define another allocation $q^\star$ by:

$$q^\star\left(l_1;k,p\right)=q\left(l_1;k,p\right)-\sum_{l_2\in L}\mathbb{1}_{\Delta(l_1,l_2;k,p)>0\text{ or }l_2\succ_p l_1}\cdot\Delta^\star\left(l_1,l_2;k,p\right)$$
$$+\sum_{l_2\in L}\mathbb{1}_{\Delta(l_2,l_1;k,p)>0\text{ or }l_2\succ_p l_1}\cdot\Delta^\star\left(l_2,l_1;k,p\right)$$

where $\Delta^\star$ is defined as:

$$\Delta^\star\left(l_1,l_2;k,p\right)=\frac{\sum_{(k',p')\in K\times\mathcal{P}(L)}\Delta\left(l_2,l_1;k',p'\right)\cdot\mu\left(k',p'\right)}{\sum_{(k',p'):l_1\succ_{p'}l_2\text{ and }\Delta(l_1,l_2;k',p')=0}\mu\left(k',p'\right)+\sum_{(k',p')\in K\times\mathcal{P}(L)}\Delta\left(l_2,l_1;k',p'\right)\cdot\mu\left(k',p'\right)}$$

Note that the flows from q to q' and the flows from q to $q^\star$ are the same since the previous system for Δ is still satisfied.

Also, $q^\star$ still dominates q since,

$$\begin{cases}\Delta^\star\left(l_1,l_2;k,p\right)>0 & \text{if } l_1\succ_p l_2\\ \Delta^\star\left(l_1,l_2;k,p\right)<\Delta\left(l_1,l_2;k,p\right) & \text{if } l_2\succ_p l_1\end{cases}$$

And $q^\star$ is envy-free since,

$$\begin{cases}\Delta^\star\left(l_1,l_2;k,p\right)\geqslant\Delta^\star\left(l_1,l_2;k,p'\right)\text{ for every }p' & \text{if } l_1\succ_p l_2\\ \Delta^\star\left(l_1,l_2;k,p\right)\geqslant 0 & \text{if } l_2\succ_p l_1\end{cases}$$

Therefore, $q^\star$ is an envy-free Pareto improvement to q. □

Proof of Lemma 6: Consider any sequence of allocations $\{q_n\}_{n=1}^{\infty}$ and the element-wise limit $q^\star$.

Fix any two students (k,p) and (k,p'), since q_n are envy-free for each n,

$$\sum_{l \succ_p l'} q_n\left(l;k,p'\right) \leqslant \sum_{l \succ_p l'} q_n\left(l;k,p\right) \;\; \forall\; l' \in L,$$

where l_s is the s -th school in the preference ranking of student (k,p).

Then,

$$\lim_{n\to\infty} \sum_{l \succ_p l'} q_n\left(l;k,p'\right) \leqslant \lim_{n\to\infty} \sum_{l \succ_p l'} q_n\left(l;k,p\right) \;\; \forall\; l' \in L,$$

$$\sum_{l \succ_p l'} q^\star\left(l;k,p'\right) \leqslant \sum_{l \succ_p l'} q^\star\left(l;k,p\right) \;\; \forall\; l' \in L.$$

Therefore, $q^\star$ is envy-free. The set is closed under limits.

Similarly, the set of Pareto improvements of any allocation is closed. □

Proof of Proposition 2: Let c, c' be two vectors of subcapacities, and q, q' be the probabilistic allocation with subcapacities c, c' respectively.

Consider allocation $q_0 = (\alpha)\, q + (1-\alpha)\, q'$ and the welfare of allocation q_0 is $(\alpha)\, W(c) + (1-\alpha)\, W(c')$

If q_0 can be obtained from PS with capacities $(\alpha)\, c + (1-\alpha)\, c'$, then $(\alpha)\, W(c) + (1-\alpha)\, W(c') = W\left((\alpha)\, c + (1-\alpha)\, c'\right)$.

Suppose q_0 is obtained from probabilistic serial, and since q_0 is envy-free from Lemma 4, q_0 is not efficient by Proposition 1.

Let V be the set of envy-free allocations that Pareto dominates $q_{0.}$

V is bounded since the set of allocations is bounded and the set of all envy-free allocations and the set of allocations that are Pareto improvements to q_0 are closed by Lemma 8. Then, V is an intersection of two compact sets implying that V is compact.

Therefore, there is an allocation $q^\star \in V$ that maximizes $W(\cdot)$.

Note that $q^\star$ must be efficient because if not, by Lemma 5, there is a envy-free Pareto improvement of $q^\star$ in V which contradicts the definition that $q^\star$ maximizes $W(\cdot)$.

$q^\star$ is envy-free and efficient, implying that $q^\star$ is the probabilistic serial allocation with capacity $(\alpha)\, c + (1-\alpha)\, c'$.

Therefore, $(\alpha)\, W(c) + (1-\alpha)\, W(c') \leqslant W((\alpha)\, c + (1-\alpha)\, c')\,, W$ is concave in c.

The function W(c) is non-decreasing for c_l^k due to the assumption that $u(\cdot) \geqslant 0$. □

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