School Allocation with Observable Characteristics

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Question

- School Allocation Problem
- Deferred Acceptance (DA) with school priority to local students (or students with siblings)
- School priorities are designed by the social planner
- Why such priorities?
- Why DA not random priority or any other mechanism?

Answer

- Maybe planer knows something about students' preferences, e.g. transportation cost,
- Students have observable characteristics, e.g. neighborhood, race, caste, tribe

Properties

- We restrict to ordinal mechanisms because it is easier for the students who report ordinal preferences
- We restrict to mechanisms that satisfy the following properties:

1. Standard: envy-freeness, efficiency
2. New: group symmetry (relaxing symmetry)

- We want to find the mechanism that maximizes expected cardinal utilities

Results and Conjectures

- We show that such symmetric, envy-free and efficient mechanism exist and is given by a modified version of Probabilistic Serial (PS)
- We would like to find conditions under which this mechanism is equivalent to the DA algorithm with priorities

Model

- L schools and K characteristics
- For example, if the characteristics is district the student lives in, then $K = \mathrm{L}$
- c_l is the capacity for school l
- μ_k is the amount of students with characteristics k
- a profile of preferences is a distribution μ where $\mu(k, p)$ is the fraction of students who have characteristics k and preference p

Ordinal Preferences

- Preferences p are permutations of $\{1, 2, ..., L\}$ (i.e. they are strict), call the set of preferences $\mathcal{P}(L)$
- Preference profiles have full support (i.e. for each k and $p, \mu(k, p) > 0$)

Cardinal utilities

- Students have cardinal utilities
- The utility distribution f_k needs to be consistent with the preference profile μ:

$$\int_{p(u)=p} f_k\left(u\right) du = \frac{\mu\left(k, p\right)}{\mu_k}$$

where $p\left(u\right)$ is the ordinal preference induced by utility $u = \left(u_1, u_2, u_3 ..., u_L\right)$

Allocations and FOSD

- An allocation is a lottery over schools $q \in \Delta(L)$:

$$\sum_{l} q(l) = 1$$

- Preferences over allocations are given by First Order Stochastic Dominance:

$$q \succsim^{FOSD} q' \text{ if } \forall\, l, \sum_{l' \succsim_s l} q(l') \geq \sum_{l' \succsim_s l} q'(l')$$

$$q \succ^{FOSD} q' \text{ if } \forall\, l, \sum_{l' \succsim_s l} q(l') > \sum_{l' \succsim_s l} q'(l')$$

Mechanisms

- A mechanism is a function $q : I \to \Delta(L)$ that maps each student from the set of all students I to an allocation

Abuse of notation: $q(l; i)$ is both the allocation and the mechanism that assigns this allocation to student i

- An mechanism is feasible if:

$$\int q(l; i)\, d\mu \leq c_l$$

Abuse of notation: $\mu(i)$ is the distribution of students

Properties

- Formal definitions:

1. Group symmetry: $q(l; i)$ only depends on group k and reported preference p, not on identity
2. Envy-free: there does not exist j such that $q(j) \succ_i^{FOSD} q(i)$, where i, j both have characteristics k
3. Efficient: there does not exist allocation q' such that $q'(i) \succsim_i^{FOSD} q(i)$ $\forall$ i, and $q'(i) \succ_i^{FOSD} q(i)$ for i in a set of students with positive measure

Notation

- Since we restrict to mechanisms that are group symmetric, define a mechanism as:

$$q : K \times \mathcal{P}(L) \to \Delta(L)$$

$$\text{such that } \sum_{k,p} q(l; k, p)\, \mu(k, p) \le c_l$$

Welfare Maximization

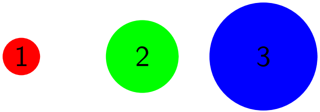
- The total welfare is the sum of expected utilities
- An allocation is optimal if it maximizes the total welfare

$$W(q) = \sum_{k=1}^{K} \int uq(l; k, p(u)) \mu(k, p(u)) f_k(u) du$$

PS Mechanism

- Each school has size c_l
- Each student eats his or her favorite among the remaining (not completely eaten) schools at rate 1
- The amount of school l eaten by a student with preference p at time 1 is the probability he or she gets allocated to school l

PS Example



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Bogomolnaia and Moulin (2001)

- Introduced the PS mechanism
- Characterized the mechanism by symmetry, envy-freeness and efficiency
- Any PS allocation is symmetric, envy-free and efficient
- Not all symmetric, envy-free and efficient allocation can be obtained by PS

Liu and Pycia (2014)

- Added the full support condition
- All full support, envy-free and efficient allocation can be obtained by PS

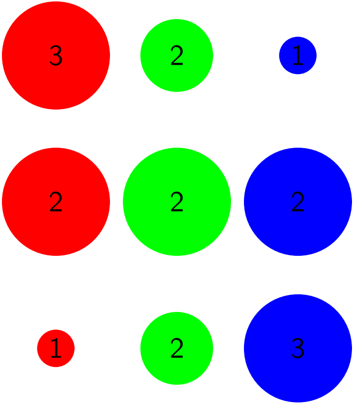
Modified PS Mechanism

- First assign capacities to group k : c_l^k such that

$$\sum_{k=1}^{K} c_l^k = c_l \ \forall \ l \in L$$
$$\sum_{l=1}^{L} c_l^k = \mu_k \ \forall \ k \in K$$

- Then run PS for each group separately

Modified PS Example



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Characterization of PS

Proposition

A mechanism is group symmetric, envy-free and efficient if and only if it is obtained by the modified PS for some capacities

- Group symmetry is implied by the definition $q(l; k, p)$
- PS $\Rightarrow$ envy-free and efficient (similar arguments from Bogomolnaia and Moulin (2001))
- envy-free and efficient $\Rightarrow$ PS (similar arguments from Liu and Pycia (2014))

Welfare function

- We restrict our attention to symmetric, envy-free and efficient mechanisms (i.e. PS with group capacities)
- Let $q^{PS}(c)$ be the allocation obtained by PS with group capacities c:

$$\max_{q} W(q) = \max_{c} W\left(q^{PS}(c)\right)$$

Properties of Welfare function

- Define the welfare for group k:

$$W_k(q) = \int uq(l; k, p(u))\,\mu(k, p(u))\, f_k(u)\, du$$

- The welfare maximization is:

$$\max_c \sum_{k=1}^{K} W_k\left(q^{PS}(c)\right) \text{ such that } \sum_k c_l^k \leq c_l \text{ and } \sum_l c_l^k \geq \mu_k$$

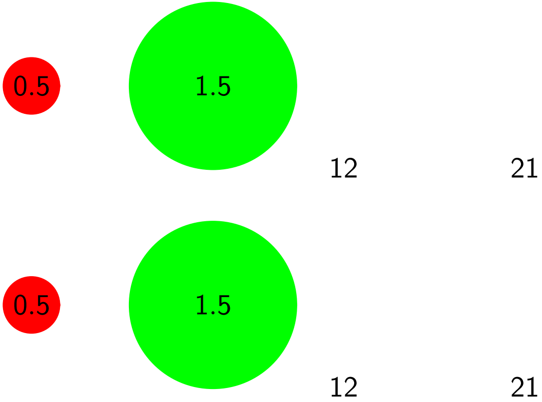
Proposition

$W_k\left(q^{PS}(c)\right)$ is non-decreasing and concave in c

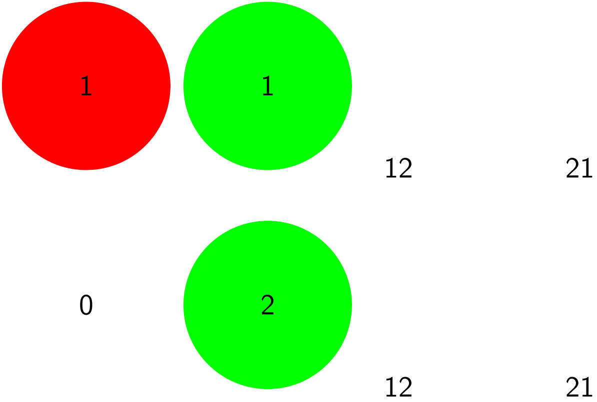
Two school case

- Only two schools and two characteristics (local students and non-local students)
- The welfare functions are piecewise linear, non-decreasing and concave in individual group capacities
- The resulting allocation is the same as the DA allocation

Two school example, equal capacity



Two school example, optimal capacity



DA Mechanism

- Every student proposes to his or her favorite school that has not rejected him or her
- The schools temporarily reject equal fraction of students with less preferred characteristics

Modified DA Mechanism

- We need to modify the algorithm for continuum of students
- Keep track of the fraction of students rejected with each characteristics
- At the beginning of each round, reject the same fraction of students who propose to this school for the first time

Two-school example

- The modification is not needed for the $K = L = 2$ case
- The resulting allocation is the same as the PS allocation with optimal capacities

Still working on showing

- Either DA can generate optimal capacities or DA is not optimal
- Characterize the utility distributions such that DA is equivalent to PS with optimal capacities
- Other suggestions?